ON VALUES OF THE BESSEL FUNCTION FOR GENERIC REPRESENTATIONS OF FINITE GENERAL LINEAR GROUPS

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Abstract. We find a recursive expression for the Bessel function of Gel'fand for irreducible generic representations of finite general linear groups. We show that certain values of the Bessel function are given as traces of exterior powers of generalized Kloosterman sheaves of Katz. As an application, we show that certain polynomials, having values of the Bessel function as their coefficients, have all of their roots lying on the unit circle.

1. Introduction

Let $\mathbb{F}$ be a finite field with q elements, let $\psi\colon \mathbb{F} \to \mathbb{C}^\times$ be a non-trivial additive character. For an irreducible generic representation π of $\mathrm{GL}_n(\mathbb{F})$, Gel'fand defined its Bessel function $\mathcal{J}_{\pi,\psi}$ in [10, Section 4], and gave a formula

$$\mathcal{J}_{\pi,\psi}(g) = \frac{1}{|U_n|} \sum_{u \in U_n} \psi^{-1}(u) \operatorname{tr}(\pi(gu)),$$

where U_n is the standard unipotent radical of $\mathrm{GL}_n(\mathbb{F})$. Gel'fand also described the support of the Bessel function. Computation of values of the Bessel function on its support using this formula is in general a hard task. Previous works include computations of the Bessel function for cuspidal representations of GL_2 [3], generic representations of GL_2 [9, 22], cuspidal representations of GL_3 [10, 3], generic representations of GL_3 [13], cuspidal representations of GL_4 [11, 8], generic representations of GL_4 [24]. These computations are involved and it is a challenge to generalize them for larger n. A formula for the values of the Bessel function for $(n-1,1)$ anti-diagonal scalar block matrices for cuspidal representations of GL_n [28, 20] and for generic representations of GL_n [6] is known.

The Bessel function is closely related to gamma factors of the representations, see [23, 19] for its relation to the Rankin-Selberg gamma factors, and [30] for its relation to the exterior square gamma factors. The Bessel function served as a key ingredient in Nien's proof of Jacquet's conjecture [19] for finite fields. The ideas of Nien were later used by Chai for the proof of Jacquet's conjecture over p-adic fields [4], where the key ingredient is an analog of the Bessel function.

In a recent work [31], Rongqing Ye and the author were able to express the Rankin-Selberg gamma factors of cuspidal irreducible representations explicitly as products of Gauss sums involving the characters parameterizing the representations. We are able to use this result in order to find a recursive expression for the Bessel function, in terms of partitions of the sizes of the relevant anti-diagonal scalar block matrices. Our technique avoids computations of the characters of the representations, which avoids Green polynomials [12] and conjugacy classes difficulties.

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We are able to find a relation between values of the Bessel function and Katz's generalized Kloosterman sheaves. Kloosterman sheaves were introduced by Deligne in [7] and studied extensively by Katz [14]. They are used in order to bound exponential sums arising in analytic number theory, and have important applications. Our computation relates values of the Bessel function of the form $\mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ cI_m & \end{pmatrix}$ to the m-th exterior power of the Frobenius action on a geometric stalk of a certain generalized Kloosterman sheaf. This is a surprising relation between representation theory of finite groups of Lie type and Kloosterman sheaves. This result can be stated as follows:

Theorem 1.1. *Let π be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$, associated with a regular character $\alpha\colon \mathbb{F}_n^\times \to \mathbb{C}^\times$. Let ℓ be a prime different than the characteristic of $\mathbb{F}$. Fix an embedding $\overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$. Consider the following generalized Kloosterman sheaf of Katz $\mathcal{K} = \mathrm{Kl}(\mathbb{F}_n, \alpha^{-1}, \psi) = \mathbf{R}\mathrm{Norm}_!(\mathrm{Trace}^* \mathrm{AS}_\psi \otimes \mathcal{L}_{\alpha^{-1}})[n-1]$, associated with the diagram*

$$\begin{array}{ccccc} & & \mathrm{Res}_{\mathbb{F}_n/\mathbb{F}}\,\mathbb{G}_m & & \\ & \swarrow^{\mathrm{Norm}} & & \searrow^{\mathrm{Trace}} & \\ \mathbb{G}_m & & & & \mathbb{A}^1 \end{array}.$$

Then for any $0 \le m \le n$, and any $c \in \mathbb{F}^\times$

$$q^{-\frac{m(n-1)}{2}} \operatorname{tr} \wedge^m \left(\mathrm{Fr}_{(-1)^{n-1}c^{-1}} \mid \mathcal{K}_{(-1)^{n-1}c^{-1}}\right) = q^{\frac{m(n-m)}{2}} \mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ cI_m & \end{pmatrix},$$

where $\wedge^m$ is the m-th exterior power, and $\mathrm{Fr}_{(-1)^{n-1}c^{-1}} \mid \mathcal{K}_{(-1)^{n-1}c^{-1}}$ is the action of the geometric Frobenius at $(-1)^{n-1}c^{-1}$ acting on the stalk of $\mathcal{K}$.

Although it is known that character sheaves are closely related to representation theory of finite groups of Lie type [2], we are not aware of any direct relation between Kloosterman sheaves and representation theory of finite groups of Lie type in the literature. We hope that this opens a door to more work on generalizations of this phenomenon, either for other groups, or for other geometric interpretations of values of the Bessel function with more than two blocks.

As a result of the above mentioned relation, we are able to show that certain polynomials whose coefficients are values of the Bessel function have the property that all of their roots lie on the unit circle.

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2. Generic representations and their local constants

In this section, we describe the irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$ in terms of Green's parametrization [12]. Then we briefly review the theories of the Rankin-Selberg gamma factors and of the tensor product ϵ_0 factors for these representations.

2.1. Parametrization of irreducible representations of $\mathrm{GL}_n(\mathbb{F})$.

We briefly review the parametrization of irreducible representations of $\mathrm{GL}_n(\mathbb{F})$, as described in [12] or [16, Section 1].

Let $\mathbb{F}$ be a field with q elements. Let $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$. For each positive integer d, we denote by $\mathbb{F}_d$ the (unique) field extension of $\mathbb{F}$ of degree d in $\overline{\mathbb{F}}$. We denote by $\widehat{\mathbb{F}_d^\times}$ the character group of $\mathbb{F}_d^\times$, the multiplicative group of $\mathbb{F}_d$. A character $\alpha \in \widehat{\mathbb{F}_d^\times}$ is called regular if the set $\{\alpha, \alpha^q, \dots, \alpha^{q^{d-1}}\}$ is of size d.

For each $d' \mid d$, we have the norm map $\mathrm{N}_{d:d'} \colon \mathbb{F}_d^\times \to \mathbb{F}_{d'}^\times$. These maps induce maps $\widehat{\mathrm{N}_{d:d'}} \colon \widehat{\mathbb{F}_{d'}^\times} \to \widehat{\mathbb{F}_d^\times}$, by mapping $\gamma \in \widehat{\mathbb{F}_{d'}^\times}$ to $\gamma \circ \mathrm{N}_{d:d'} \in \widehat{\mathbb{F}_d^\times}$. We have that $\left(\widehat{\mathbb{F}_d^\times}\right)_d$ with the norm maps $\left(\widehat{\mathrm{N}_{d:d'}}\right)_{d' \mid d}$ forms a directed system. We denote its direct limit $\Gamma = \varinjlim \widehat{\mathbb{F}_d^\times}$.

Let $\mathrm{Fr} \in \mathrm{Gal}\left(\overline{\mathbb{F}}/\mathbb{F}\right)$ be the geometric Frobenius automorphism, i.e., $\mathrm{Fr}(x^q) = x$, for every $x \in \overline{\mathbb{F}}$. Then Fr acts on Γ by $\mathrm{Fr}\,\gamma = \gamma^q$. We identify $\widehat{\mathbb{F}_d^\times}$ with the subgroup $\Gamma_d = \left\{\gamma \in \Gamma \mid \mathrm{Fr}^d \gamma = \gamma\right\}$ of Γ. If f is a Frobenius orbit, i.e., a set of the form $f = \left\{\mathrm{Fr}^i \gamma \mid i \in \mathbb{Z}\right\}$ for $\gamma \in \Gamma$, we define its degree $d(f)$ to be its cardinality. Then for $\gamma \in f$, we have $\gamma \in \Gamma_{d(f)}$. We denote by $\mathrm{Fr} \backslash \Gamma$ the set of Frobenius orbits.

A partition of an integer $n \geq 0$, is a tuple of integers $\lambda = (n_1, \dots, n_r)$, with $n_1 \geq n_2 \geq \dots \geq n_r > 0$, and such that $n_1 + \dots + n_r = n$. We denote by $|\lambda| = n$, the size of λ, we denote by $\mathfrak{n}(\lambda) = r$, the length of λ. We write $\lambda \vdash n$ to specify that λ is a partition of n. We denote by $()$ the empty partition of 0. We denote by $\mathcal{P}$ the set of all partitions of all non-negative integers.

Denote by $P_n(\Gamma)$ the set of partition valued functions $\phi \colon \Gamma \to \mathcal{P}$, such that:

(1) $\phi \circ \mathrm{Fr} = \phi$, i.e., ϕ is constant on Frobenius orbits.
(2) $\sum_{\gamma \in \Gamma} |\phi(\gamma)| = n$.

If f is a Frobenius orbit, then such ϕ is constant on the elements of f, and we define $\phi(f) = \phi(\gamma)$, where $\gamma \in f$.

For representations $\pi_1, \dots, \pi_r$ of $\mathrm{GL}_{n_1}(\mathbb{F}), \dots, \mathrm{GL}_{n_r}(\mathbb{F})$ respectively, we denote their parabolic induction, a representation of $\mathrm{GL}_{n_1 + \dots + n_r}(\mathbb{F})$, by $\pi_1 \circ \dots \circ \pi_r$. This operation is commutative and associative.

We now describe the irreducible representations of $\mathrm{GL}_n(\mathbb{F})$. These are in a bijection with the set $P_n(\Gamma)$. First, let f be a Frobenius orbit of degree d, then f corresponds to an irreducible cuspidal representation Π_f of $\mathrm{GL}_d(\mathbb{F})$. For any positive integer s, consider the representation of GL_{ds} defined by $\Pi_f^{\circ s} = \Pi_f \circ \dots \circ \Pi_f$, where $\circ$ is performed s times. Then the irreducible subrepresentations of $\Pi_f^{\circ s}$ are indexed by partitions of s. For each partition λ of s, we denote by Π_f^λ the irreducible subrepresentation of $\Pi_f^{\circ s}$ corresponding to λ.

Let $\phi \in P_n(\Gamma)$. Suppose that ϕ is supported on the pairwise distinct Frobenius orbits $f_1, \dots, f_t$, i.e., $\phi(\gamma) \neq ()$ if and only if $\gamma \in f_i$ for some $1 \leq i \leq t$. We associate to ϕ the representation $\Pi_\phi = \Pi_{f_1}^{\phi(f_1)} \circ \dots \circ \Pi_{f_t}^{\phi(f_t)}$.

Theorem 2.1. *The map $\phi \mapsto \Pi_\phi$ is a bijection between $P_n(\Gamma)$ and the set of equivalence classes of irreducible representations of* $\mathrm{GL}_n(\mathbb{F})$.

2.2. **Parametrization of irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$.** Let $\psi \colon \mathbb{F} \to \mathbb{C}^\times$ be a non-trivial additive character. Let U_n be the upper unipotent subgroup of $\mathrm{GL}_n(\mathbb{F})$. Then ψ defines a character on U_n by

$$\psi \begin{pmatrix} 1 & a_1 & * & * & * \\ & 1 & a_2 & * & * \\ & & \ddots & \ddots & * \\ & & & 1 & a_{n-1} \\ & & & & 1 \end{pmatrix} = \psi\left(\sum_{i=1}^{n-1} a_i\right).$$

A representation π of $\mathrm{GL}_n(\mathbb{F})$ is called generic if $\mathrm{Hom}_{U_n}(\pi \restriction_{U_n}, \psi) \neq 0$, or equivalently by Frobenius reciprocity, if $\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{F})}\left(\pi, \mathrm{Ind}_{U_n}^{\mathrm{GL}_n(\mathbb{F})}(\psi)\right) \neq 0$. This condition does not depend on the choice of the non-trivial additive character ψ.

It is known that if $\pi_1, \dots, \pi_r$ are representations of $\mathrm{GL}_{n_1}(\mathbb{F}), \dots, \mathrm{GL}_{n_r}(\mathbb{F})$ respectively, then $\pi_1 \circ \dots \circ \pi_r$ is generic, if and only if $\pi_1, \dots, \pi_r$ are all generic [18, Theorem 5.2.1]. Therefore, in light of Theorem 2.1, in order to classify the irreducible generic representations, it suffices to classify the irreducible generic subrepresentations of $\Pi_f^{\circ s}$, for a Frobenius orbit f and $s \geq 1$.

It is known that irreducible cuspidal representations of $\mathrm{GL}_n(\mathbb{F})$ are generic. By [26], if f is a Frobenius orbit of degree d, then for each s, the representation $\Pi_f^{\circ s}$ has a unique irreducible generic representation, which corresponds to the partition (s) of s. This is the "generalized Steinberg" representation. Its dimension is given by the formula

$$\dim \Pi_f^{(s)} = q^{\frac{ds(s-1)}{2}} \frac{\prod_{j=1}^{ds}(q^j - 1)}{\prod_{j=1}^{s}(q^{dj} - 1)}.$$

Therefore we have that irreducible generic representations are parameterized by $\phi \in P_n(\Gamma)$, such that for every $\gamma \in \Gamma$, $\phi(\gamma) = ()$ or $\phi(\gamma)$ is of the form (s), where s is a positive integer.

Suppose that $\phi \in P_n(\Gamma)$ parameterizes an irreducible generic representation as above and is supported on the Frobenius orbits $f_1, \dots, f_t$ of degrees $d_1 = d(f_1), \dots, d(f_t) = d_t$, and that for every i, $\phi(f_i) = (s_i)$. Then

$$\dim \Pi_\phi = \frac{|\mathrm{GL}_n(\mathbb{F})|}{\frac{|U_n|}{\prod_{i=1}^t |U_{s_i d_i}|} \prod_{i=1}^t |\mathrm{GL}_{s_i d_i}(\mathbb{F})|} \cdot \prod_{i=1}^t \dim \Pi_{f_i}^{(s_i)} = [\mathrm{GL}_n(\mathbb{F}) : U_n] \prod_{i=1}^t \frac{q^{\frac{d_i s_i (s_i - 1)}{2}}}{\prod_{j=1}^{s_i}(q^{d_i j} - 1)}.$$

We note that one can give an alternative definition for Π_ϕ by defining it to be the unique irreducible generic representation of $\Pi_{f_1}^{\circ s_1} \circ \dots \circ \Pi_{f_r}^{\circ s_r}$.

Recall that irreducible cuspidal representations of $\mathrm{GL}_n(\mathbb{F})$ are in a bijection with monic irreducible polynomials of degree n, not having zero as a root. By factorization of polynomials of degree n into powers of irreducible polynomials, we get that the irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$ are in a bijection with monic polynomials that don't vanish at zero. Therefore there exist exactly $q^n - q^{n-1}$ irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$.

2.3. **Whittaker models and their Bessel functions.** Recall that the representation $\mathrm{Ind}_{U_n}^{\mathrm{GL}_n(\mathbb{F})}(\psi)$ is multiplicity free [10, Theorem 0.5], i.e., for every irreducible representation π of $\mathrm{GL}_n(\mathbb{F})$, we have that $\dim \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{F})}\left(\pi, \mathrm{Ind}_{U_n}^{\mathrm{GL}_n(\mathbb{F})}(\psi)\right) \leq 1$. If π is generic, we get that $\dim \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{F})}\left(\pi, \mathrm{Ind}_{U_n}^{\mathrm{GL}_n(\mathbb{F})}(\psi)\right) = 1$, and we denote by $\mathcal{W}(\pi,\psi)$ the unique subspace of $\mathrm{Ind}_{U_n}^{\mathrm{GL}_n(\mathbb{F})}(\psi)$ that is isomorphic to π, that is the *Whittaker model of* π*, with respect to the additive character* ψ.

Suppose that π is an irreducible generic representation of $\mathrm{GL}_n(\mathbb{F})$. There exists a special element of $\mathcal{W}(\pi,\psi)$, denoted by $\mathcal{J}_{\pi,\psi}$, which is called the *normalized Bessel function of* π*, with respect to the additive character* ψ. It is the unique element of $\mathcal{W}(\pi,\psi)$ satisfying the following properties:

(1) $\mathcal{J}_{\pi,\psi}(I_n) = 1$.
(2) $\mathcal{J}_{\pi,\psi}(u_1 g u_2) = \psi(u_1)\psi(u_2)\mathcal{J}_{\pi,\psi}(g)$, for every $u_1, u_2 \in U_n$ and $g \in \mathrm{GL}_n(\mathbb{F})$.

For $n_1, \dots, n_s > 0$ with $n = n_1 + \dots + n_s$ and $c_1, \dots, c_s \in \mathbb{F}^\times$, denote

$$g_{n_1,\dots,n_s}(c_1,\dots,c_s) = \begin{pmatrix} & & & c_1 I_{n_1} \\ & & c_2 I_{n_2} & \\ & \iddots & & \\ c_s I_{n_s} & & & \end{pmatrix}.$$

A counting argument shows that there exist exactly $q^n - q^{n-1}$ options for a matrix of the form $g_{n_1,\dots,n_s}(c_1,\dots,c_s) \in \mathrm{GL}_n(\mathbb{F})$, for any choice of s, $n_1,\dots,n_s > 0$ with $n_1 + \dots + n_s = n$ and any choice of $c_1,\dots,c_s \in \mathbb{F}^\times$. We have the following proposition regarding the support of $\mathcal{J}_{\pi,\psi}$:

Proposition 2.2 ([10, Proposition 4.9])**.** *$\mathcal{J}_{\pi,\psi}$ is supported on double cosets of the form*

$$U_n \cdot g_{n_1,\dots,n_s}(c_1,\dots,c_s) \cdot U_n.$$

We also have the following relations between $\mathcal{J}_{\pi,\psi}$ and its complex conjugate:

Proposition 2.3 ([19, Proposition 2.15, Proposition 3.5])**.** *For any* $g \in \mathrm{GL}_n(\mathbb{F})$*:*

(1) $\mathcal{J}_{\pi,\psi}(g^{-1}) = \overline{\mathcal{J}_{\pi,\psi}(g)}$.
(2) $\mathcal{J}_{\pi^\vee,\psi^{-1}}(g) = \overline{\mathcal{J}_{\pi,\psi}(g)}$*, where* $\pi^\vee$ *denotes the contragredient representation of* π*.*

Gel'fand [10, Proposition 4.5] gives a formula for $\mathcal{J}_{\pi,\psi}$ in terms of the trace character of the representation π:

Theorem 2.4. *For any* $g \in \mathrm{GL}_n(\mathbb{F})$*,*

$$\mathcal{J}_{\pi,\psi}(g) = \frac{1}{|U_n|}\sum_{u \in U_n} \psi^{-1}(u)\,\mathrm{tr}(\pi(gu)).$$

2.4. **Rankin-Selberg gamma factors.** Let $n \geq m$ be integers and let π, σ be irreducible generic representations of $\mathrm{GL}_n(\mathbb{F}), \mathrm{GL}_m(\mathbb{F})$ respectively. The Rankin-Selberg gamma factor $\gamma(\pi \times \sigma, \psi)$ was defined in Piatetski-Shapiro's unpublished lecture notes from 1976. It was also defined for $m < n$ in Roditty's master's thesis, under the supervision of David Soudry [23]. The main ideas of Roditty's thesis are covered by Nien in [19]. We briefly review the main results that we need.

The first result is the functional equation, which defines the Rankin-Selberg gamma factors.

Theorem 2.5. *Suppose that $n > m$, and that π is cuspidal. Then there exists a non-zero constant $\gamma(\pi \times \sigma, \psi) \in \mathbb{C}^\times$, such that for every $0 \le k \le n - m - 1$, every $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\sigma, \psi^{-1})$,*

$$q^{mk}\gamma(\pi \times \sigma, \psi) \sum_{h \in U_m \backslash \mathrm{GL}_m(\mathbb{F})} \sum_{x \in M_{(n-m-k-1)\times m}(\mathbb{F})} W\begin{pmatrix} h & & \\ x & I_{n-m-k-1} & \\ & & I_{k+1} \end{pmatrix} W'(h) = \sum_{h \in U_m \backslash \mathrm{GL}_m(\mathbb{F})} \sum_{x \in M_{m\times k}(\mathbb{F})} W\begin{pmatrix} & I_{n-m-k} & \\ & & I_k \\ h & & x \end{pmatrix} W'(h).$$

Piatetski-Shapiro proved in his lecture the functional equation for the case $n = m$ [29, Theorem 2.3]:

Theorem 2.6. *Suppose that $n = m$, π, σ are cuspidal. There exists a non-zero constant $\gamma(\pi \times \sigma, \psi)$, such that for every $W \in \mathcal{W}(\pi, \psi), W' \in \mathcal{W}(\sigma, \psi^{-1})$, and for every $\phi \colon \mathbb{F}^n \to \mathbb{C}$ with $\phi(0) = 0$, the following functional equation holds:*

$$\gamma(\pi \times \sigma, \psi) \sum_{g \in U_n \backslash \mathrm{GL}_n(\mathbb{F})} W(g) W'(g) \phi(e_n g) = \sum_{g \in U_n \backslash \mathrm{GL}_n(\mathbb{F})} W(g) W'(g) \mathcal{F}_\psi \phi\left(e_1 {}^t g^{-1}\right),$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{F}^n$, $e_n = (0, \dots, 0, 1) \in \mathbb{F}^n$ and $\mathcal{F}_\psi \phi$ is the Fourier transform of ϕ given by

$$\mathcal{F}_\psi \phi(x) = \sum_{y \in \mathbb{F}^n} \phi(y) \psi(\langle x, y \rangle).$$

Remark 2.7. We normalize the Fourier transform like in [29], so that the tensor product ϵ_0-factors and the Rankin-Selberg gamma factors will satisfy the same relation for both $n < m$ and $n = m$. This relation will be discussed in the next sections.

We have the following expression for the Rankin-Selberg gamma factor in terms of Bessel functions.

Proposition 2.8 ([23, Lemma 6.1.4] or [19, Proposition 2.16])**.** *Suppose $n > m$, π cuspidal. Then*

$$\gamma(\pi \times \sigma, \psi) = \sum_{g \in U_m \backslash \mathrm{GL}_m(\mathbb{F})} \mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ g & \end{pmatrix} \mathcal{J}_{\sigma,\psi^{-1}}(g). \tag{1}$$

We also have an expression for the case $m = n$:

Proposition 2.9 ([29, eq. 16])**.** *Suppose $m = n$, then*

$$\gamma(\pi \times \sigma, \psi) = \sum_{g \in U_n \backslash \mathrm{GL}_n(\mathbb{F})} \mathcal{J}_{\pi,\psi}(g) \mathcal{J}_{\sigma,\psi^{-1}}(g) \psi\left(e_1 {}^t g^{-1} {}^t e_n\right), \tag{2}$$

where $e_1 = (1, 0, \dots, 0)$, $e_n = (0, \dots, 0, 1)$.

In the case where $\pi \cong \sigma^\vee$, we actually have by [29, Corollary 4.3] $\gamma(\pi \times \pi^\vee, \psi) = -1$.

2.5. **Langlands-Shahidi gamma factors.** Let π, σ be irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$, $\mathrm{GL}_m(\mathbb{F})$ respectively.

In an unpublished note from 1979 [27], Soudry defines an intertwining operator $U\colon \sigma\circ\pi \to \pi\circ\sigma$, which allows him to define a gamma factor, which he denotes $\Gamma_{\pi,\sigma}(1)$. This is a finite field analog of the Langlands-Shahidi gamma factor.

This gamma factor can be defined for any pair of irreducible generic representations π, σ, regardless whether $n > m$ or whether π, σ are cuspidal. The gamma factor satisfies

$$\Gamma_{\pi,\sigma}(1) = \omega_\pi(-1)^m \omega_\sigma(-1)^n \Gamma_{\sigma^\vee,\pi^\vee}(1),$$

where ω_π, ω_σ are the central characters of π, σ correspondingly.

Soudry expresses $\Gamma_{\pi,\sigma}(1)$ in terms of the associated Bessel functions:

Theorem 2.10. *(1) If $n > m$, then*

$$\Gamma_{\pi,\sigma}(1) = q^{\frac{m}{2}(2n-m-1)}\omega_\sigma(-1) \sum_{g\in U_m\backslash \mathrm{GL}_m(\mathbb{F})} \mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ g & \end{pmatrix} \mathcal{J}_{\sigma^\vee,\psi^{-1}}(g).$$

(2) If $n = m$, then

$$\Gamma_{\pi,\sigma}(1) = q^{\frac{n(n-1)}{2}}\omega_\sigma(-1) \sum_{g\in U_n\backslash \mathrm{GL}_n(\mathbb{F})} \psi\begin{pmatrix} I_n & g^{-1} \\ & I_n \end{pmatrix} \mathcal{J}_{\pi,\psi}(g)\, \mathcal{J}_{\sigma^\vee,\psi^{-1}}(g).$$

Then Soudry shows that these gamma factors are multiplicative:

Theorem 2.11 (Multiplicativity of gamma factors)**.** *Let π,σ be irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$, $\mathrm{GL}_m(\mathbb{F})$ respectively. Suppose that σ_1, σ_2 are irreducible generic representations of $\mathrm{GL}_{m_1}(\mathbb{F})$, $\mathrm{GL}_{m_2}(\mathbb{F})$ respectively, such that $m_1 + m_2 = m$, and suppose that $\sigma \subseteq \sigma_1 \circ \sigma_2$. Then*

$$\Gamma_{\pi,\sigma}(1) = \Gamma_{\pi,\sigma_1}(1)\,\Gamma_{\pi,\sigma_2}(1).$$

We notice that if π is cuspidal and $n < m$, then by Proposition 2.8 and Theorem 2.10

$$\gamma(\pi\times\sigma,\psi) = \omega_\sigma(-1)\, q^{-\frac{m}{2}(2n-m-1)}\Gamma_{\pi,\sigma^\vee}(1),$$

and this formula also holds for $n = m$, if π, σ are cuspidal by Proposition 2.9 and Theorem 2.10.

We extend our definition of $\gamma(\pi\times\sigma,\psi)$ to all irreducible generic representations, by defining $\gamma(\pi\times\sigma,\psi) = \omega_\sigma(-1)\, q^{-\frac{m}{2}(2n-m-1)}\Gamma_{\pi,\sigma^\vee}(1)$. Then under this notation we have:

Theorem 2.12 (Soudry)**.** *Let π, σ be irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$, $\mathrm{GL}_m(\mathbb{F})$ respectively. Let ω_π, ω_σ denote the central characters of π, σ respectively.*

(1) If $n > m$, then $\gamma(\pi\times\sigma,\psi)$ is given by eq. (1).
(2) If $n = m$, then $\gamma(\pi\times\sigma,\psi)$ is given by eq. (2).
(3)

$$\gamma(\pi\times\sigma,\psi) = q^{\frac{m^2+m}{2}-\frac{n^2+n}{2}}\omega_\pi(-1)^{m-1}\omega_\sigma(-1)^{n-1}\gamma(\sigma\times\pi,\psi).$$

(4) Let $m = m_1 + m_2$, and let σ_1, σ_2 be irreducible generic representations of $\mathrm{GL}_{m_1}(\mathbb{F})$, $\mathrm{GL}_{m_2}(\mathbb{F})$ respectively. Suppose σ is the unique irreducible generic subrepresentation of the parabolic induction $\sigma_1\circ\sigma_2$. Then

$$\gamma(\pi\times\sigma,\psi) = q^{m_1 m_2}\gamma(\pi\times\sigma_1,\psi)\,\gamma(\pi\times\sigma_2,\psi).$$

(5) Let $n = n_1 + n_2$, and let π_1, π_2 be irreducible generic representations of $\mathrm{GL}_{n_1}(\mathbb{F})$, $\mathrm{GL}_{n_2}(\mathbb{F})$ respectively. Suppose π is the unique irreducible generic subrepresentation of the parabolic induction $\pi_1 \circ \pi_2$. Then

$$\gamma(\pi \times \sigma, \psi) = q^{-\frac{m^2+m}{2}} \omega_\sigma(-1)\, \gamma(\pi_1 \times \sigma, \psi)\, \gamma(\pi_2 \times \sigma, \psi).$$

2.6. **A recursive expression for the Bessel function.** Using Theorem 2.4 and the orthogonality relations of trace characters of irreducible representations, one can also show the following:

Proposition 2.13 ([29, Lemma 4.2])**.** *Let σ, σ' be two irreducible generic representations of $\mathrm{GL}_m(\mathbb{F})$. Then*

$$\sum_{g \in U_m \backslash \mathrm{GL}_m(\mathbb{F})} \mathcal{J}_{\sigma,\psi}(g)\, \mathcal{J}_{\sigma',\psi^{-1}}(g) = \begin{cases} \frac{[\mathrm{GL}_m(\mathbb{F}) : U_m]}{\dim \sigma} & \sigma^\vee \cong \sigma' \\ 0 & \textit{otherwise} \end{cases}. \tag{3}$$

We can use Proposition 2.13 and Theorem 2.10 to show:

Theorem 2.14. *Let π be an irreducible generic representation of $\mathrm{GL}_n(\mathbb{F})$, and let $g \in \mathrm{GL}_m(\mathbb{F})$. Denote*

$$\mathcal{F}_{\pi,m,\psi}(g) = \frac{1}{[\mathrm{GL}_m(\mathbb{F}) : U_m]} \sum_\sigma \dim \sigma \cdot \gamma(\pi \times \sigma^\vee, \psi) \cdot \mathcal{J}_{\sigma,\psi}(g),$$

where σ runs over all the irreducible generic representations of $\mathrm{GL}_m(\mathbb{F})$. Then

(1) If $m < n$,

$$\mathcal{F}_{\pi,m,\psi}(g) = \mathcal{J}_{\pi,\psi}\begin{pmatrix} 0 & I_{n-m} \\ g & 0 \end{pmatrix}.$$

(2) If $m = n$,

$$\mathcal{F}_{\pi,m,\psi}(g) = \mathcal{J}_{\pi,\psi}(g)\, \psi\begin{pmatrix} I_n & g^{-1} \\ & I_n \end{pmatrix}.$$

(3) If $m > n$ and $g = cI_m$ for $c \in \mathbb{F}^\times$, then $\mathcal{F}_{\pi,m,\psi}(g) = 0$.

This is similar to equation (6.15) in [23, Theorem 6.2.1], except that we have an explicit formula for the coefficients.

Proof. Denote by $\mathcal{G}_m$ the set of irreducible generic representations of $\mathrm{GL}_m(\mathbb{F})$, and denote by $\mathfrak{g}_m$ the set of matrices of the form $g_{m_1,\dots,m_s}(c_1,\dots,c_s)$ with $m_1 + \dots + m_s = m$, $m_1,\dots,m_s > 0$ and $c_1,\dots,c_s \in \mathbb{F}^\times$.

For simplicity, we first assume $m < n$. We restate Proposition 2.13 and eq. (1) in matrix form. To begin, notice that the summand of the left hand side of eq. (3) is invariant under right multiplication by elements of U_m. Therefore using Proposition 2.2, we can rewrite Proposition 2.13 as

$$\sum_{h \in \mathfrak{g}_m} c_h \cdot \mathcal{J}_{\sigma,\psi}(h)\, \mathcal{J}_{\sigma',\psi^{-1}}(h) = \begin{cases} \frac{[\mathrm{GL}_m(\mathbb{F}) : U_m]}{\dim \sigma} & \sigma^\vee \cong \sigma' \\ 0 & \text{otherwise} \end{cases}, \tag{4}$$

where for $h \in \mathfrak{g}_m$, c_h is the size of the set $\{U_m h u \mid u \in U_m\}$. One can show that if $h = g_{m_1,\dots,m_s}(c_1,\dots,c_s)$, then $c_h = q^{\binom{m}{2} - \sum_{j=1}^r \binom{m_j}{2}}$, see for instance [30, Lemma 2.29], but we won't need this. Similarly, we have that the summand in eq. (1) is invariant under right multiplication by elements of U_m, and hence we have by Theorem 2.12 that

$$\gamma\left(\pi \times \sigma, \psi\right) = \sum_{h \in \mathfrak{g}_m} c_h \cdot \mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ h & \end{pmatrix} \mathcal{J}_{\sigma,\psi^{-1}}\left(h\right). \tag{5}$$

Let $B_{m,\psi}$ be a matrix whose rows are indexed by $\mathcal{G}_m$ and whose columns are indexed by $\mathfrak{g}_m$, having the value $\mathcal{J}_{\sigma,\psi}\left(h\right)$ at position (σ, h). Then $B_{m,\psi}$ is a square matrix of size $(q^m - q^{m-1}) \times (q^m - q^{m-1})$. By Proposition 2.3, we have that its conjugate transpose $B^*_{m,\psi}$ is a matrix whose rows are indexed by $\mathfrak{g}_m$, whose columns indexed by $\mathcal{G}_m$, having the value $\mathcal{J}_{\sigma^\vee,\psi^{-1}}\left(h\right)$ at position (h, σ).

Let C_m be a diagonal square matrix whose rows and columns are indexed by $\mathfrak{g}_m$, having the value c_h as the $h \in \mathfrak{g}_m$ entry of its diagonal. Let D_m be a diagonal square matrix whose rows and columns are indexed by $\mathcal{G}_m$, having the value $\frac{\dim \sigma}{[\mathrm{GL}_m(\mathbb{F}):U_m]}$ as the σ entry of its diagonal. Then by eq. (4), we have the relation $D_m B_{m,\psi} C_m B^*_{m,\psi} = I_{\mathcal{G}_m}$, where $I_{\mathcal{G}_m}$ is the identity matrix whose rows and columns are indexed by $\mathcal{G}_m$. Therefore $D_m B_{m,\psi} C_m$ is the inverse of $B^*_{m,\psi}$, and hence we have that $B^*_{m,\psi} D_m B_{m,\psi} C_m = I_{\mathfrak{g}_m}$, where the right hand side is the identity matrix whose rows and columns are indexed by $\mathfrak{g}_m$.

Let $v_{\pi,m,\psi}$ be the column vector whose rows are indexed by $\mathcal{G}_m$, having at position σ the value $\gamma\left(\pi \times \sigma, \psi\right)$, and let $b_{\pi,m,\psi}$ the column vector indexed by $\mathfrak{g}_m$, having at position $h \in \mathfrak{g}_m$ the value $\mathcal{J}_{\pi,\psi}\left(\begin{smallmatrix} & I_{n-m} \\ h & \end{smallmatrix}\right)$. Then we have by eq. (5) that $B_{m,\psi^{-1}} C_m b_{\pi,m,\psi} = v_{\pi,m,\psi}$. Multiplying both sides by $B^*_{m,\psi^{-1}} D_m$ from the left, we get

$$b_{\pi,m,\psi} = B^*_{m,\psi^{-1}} D_m v_{\pi,m,\psi},$$

which implies that

$$\mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ h & \end{pmatrix} = \sum_{\sigma \in \mathcal{G}_m} \mathcal{J}_{\sigma^\vee,\psi}\left(h\right) \cdot \frac{\dim \sigma}{\left[\mathrm{GL}_m\left(\mathbb{F}\right) : U_m\right]} \cdot \gamma\left(\pi \times \sigma, \psi\right),$$

For any $h \in \mathfrak{g}_m$. Replacing σ with $\sigma^\vee$ and using the equivariance properties of the Bessel function, we get the desired result.

The case $m = n$ is treated similarly, by expressing the formula in eq. (2) as a linear system $B_{m,\psi^{-1}} C_m b_{\pi,m,\psi} = v_{\pi,m,\psi}$, where this time $b_{\pi,m,\psi}$ is the column vector whose rows are indexed by $\mathfrak{g}_m$, having at position $h \in \mathfrak{g}_m$ the value $\mathcal{J}_{\pi,\psi}\left(g\right) \psi \begin{pmatrix} I_n & g^{-1} \\ & I_n \end{pmatrix}$.

For the case $m > n$ and $g = cI_m$, we write eq. (1) again as a linear system and use Theorem 2.12 (3). We get zero, because $b_{\pi,m,\psi}$ this time has zeros at entries that are not of the form $\left(\begin{smallmatrix} & I_{m-n} \\ h & \end{smallmatrix}\right)$, where $h \in \mathfrak{g}_n$. □

2.7. **Epsilon factors.** In [31], Rongqing Ye and the author defined ϵ_0-factors, and in particular tensor product ϵ_0-factors. We briefly review the results we need from there.

For two irreducible cuspidal representations, their tensor product ϵ_0-factors can be described as products of Gauss sums of their corresponding characters:

Theorem 2.15 ([31, Theorem 4.2])**.** *Let f, g be Frobenius orbits of degrees n, m respectively, and let be Π_f, Π_g their corresponding irreducible cuspidal representations of $\mathrm{GL}_n\left(\mathbb{F}\right)$, $\mathrm{GL}_m\left(\mathbb{F}\right)$ respectively. Let $\alpha \in f$ and $\beta \in g$. Then*

$$\epsilon_0\left(\Pi_f \times \Pi_g, \psi\right) = \left(-1\right)^{nm} q^{-\frac{nm}{2}} \prod_{k=1}^{\gcd(n,m)} \tau\left(\alpha \circ \mathrm{N}_{\mathrm{lcm}(n,m):n} \cdot \beta^{q^{k-1}} \circ \mathrm{N}_{\mathrm{lcm}(n,m):m}, \psi_{\mathrm{lcm}(n,m)}\right),$$

where for every r and every $\gamma \in \widehat{\mathbb{F}_r^\times}$, $\tau(\gamma, \psi_r)$ is the Gauss sum

$$\tau(\gamma, \psi_r) = -\sum_{\xi \in \mathbb{F}_r^\times} \gamma^{-1}(\xi)\, \psi_r(\xi),$$

where $\psi_r = \psi \circ \mathrm{Tr}_{\mathbb{F}_r/\mathbb{F}}$.

In order to compute the tensor product ϵ_0-factors for general irreducible representations, we use the following multiplicativity property:

Theorem 2.16. *Let $\phi \in P_n(\Gamma)$, $\phi' \in P_m(\Gamma)$ parameterize irreducible representations of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$ respectively. Then*

$$\epsilon_0(\Pi_\phi \times \Pi_{\phi'}, \psi) = \prod_{f,g \in \mathrm{Fr}\backslash\Gamma} \epsilon_0(\Pi_f \times \Pi_g, \psi)^{|\phi(f)| \cdot |\phi'(g)|},$$

where $\mathrm{Fr}\backslash\Gamma$ is the set of all the Frobenius orbits of Γ.

Next, we give a relation between the Rankin-Selberg gamma factors and the tensor product ϵ_0 factors.

Theorem 2.17. *Suppose $n \geq m$. Let π, σ be irreducible cuspidal representations of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$ respectively. Denote by ω_σ the central character of σ. Then*

$$\gamma(\pi \times \sigma, \psi) = q^{-\frac{m(n-m-1)}{2}} \omega_\sigma(-1)^{n-1} \epsilon_0(\pi \times \sigma, \psi)$$

Proof. This was done for $n > m$ in [31, Theorem 4.4]. We are left to consider the case $n = m$. For $n = m$, and $\pi \not\cong \sigma^\vee$, we have an equality of the Rankin-Selberg gamma factor corresponding to $\pi \times \sigma$ and the Rankin-Selberg gamma factor corresponding to $\Pi \times \Sigma$, where Π, Σ are level zero representations constructed from π, σ respectively [29, Theorem 4.1]. Now one can proceed exactly as in the proof of [31, Theorem 4.4]. The normalization of the Fourier transform $\mathcal{F}_\psi$ contributes the constant $q^{-\frac{n(n-n-1)}{2}} = q^{\frac{n}{2}}$.

We are left to deal with the case $m = n$ and $\pi \cong \sigma^\vee$. In this case (under our normalization), $\gamma(\pi \times \pi^\vee, \psi) = -1$ [29, Corollary 4.3]. We therefore need to show that $\epsilon_0(\pi \times \pi^\vee, \psi) = -\omega_\pi(-1)^{n-1} q^{-\frac{n}{2}}$, where ω_π is the central character of π. Let $\pi = \Pi_f$, where f is a Frobenius orbit of degree n, and let $\alpha \in f$. Then by Theorem 2.15 we have

$$\epsilon_0(\pi \times \pi^\vee, \psi) = (-1)^{n^2} q^{-\frac{n^2}{2}} \prod_{i=0}^{n-1} \tau\left(\alpha \cdot \alpha^{-q^i}, \psi_n\right).$$

Notice that $\alpha^{1-q^i} = \left(\left(\alpha^{1-q^{n-i}}\right)^{-1}\right)^{q^i}$. Since $\tau(\beta^{-1}, \psi_n) = \overline{\tau(\beta, \psi_n)}\beta(-1)$ for $\beta \in \widehat{\mathbb{F}_n^\times}$, and since Gauss sums are constant for characters in the same Frobenius orbit, we have that

$$\tau\left(\alpha^{1-q^i}, \psi_n\right) \tau\left(\alpha^{1-q^{n-i}}, \psi_n\right) = \left|\tau\left(\alpha^{1-q^i}, \psi_n\right)\right|^2 \alpha^{1-q^i}(-1) = q^n.$$

For odd $n = 2m+1$, we therefore get $\epsilon_0(\pi \times \pi^\vee, \psi) = -q^{-\frac{n^2}{2}} \cdot \tau(1, \psi_n) \cdot (q^n)^{\frac{n-1}{2}} = -q^{-\frac{n}{2}} = -q^{-\frac{n}{2}} \omega_\pi(-1)^{n-1}$. For even $n = 2m$, we get $\epsilon_0(\pi \times \pi^\vee, \psi) = q^{-\frac{n^2}{2}} \cdot \tau(1, \psi_n) \cdot \tau\left(\alpha^{1-q^m}, \psi_n\right) \cdot (q^n)^{\frac{n-2}{2}} = q^{-n} \tau\left(\alpha^{1-q^m}, \psi_n\right)$. Notice that α^{1-q^m} is trivial on $\mathbb{F}_m^\times$. By Proposition A.2, $\tau\left(\alpha^{1-q^m}, \psi_{2m}\right) = -q^m \cdot \alpha^{q^m-1}(z)$, where $z \in \mathbb{F}_{2m}^\times$ satisfies $z^{q^m-1} = -1$. Then $\epsilon_0(\pi \times \pi^\vee, \psi) = -q^{-\frac{n}{2}} \alpha(-1)$, and then statement follows since $\alpha(-1) = \omega_\pi(-1) = \omega_\pi(-1)^{n-1}$. ∎

In order to relate $\gamma(\pi\times\sigma,\psi)$ to $\epsilon_0(\pi\times\sigma,\psi)$ in the general case, we use Theorem 2.17 combined with the multiplicativity property of the tensor product ϵ_0-factors (Theorem 2.16), and the properties of $\gamma(\pi\times\sigma,\psi)$ shown by Soudry (Theorem 2.12). Using these repeatedly, we get the following result:

Theorem 2.18. *Let π, σ be irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$, $\mathrm{GL}_m(\mathbb{F})$ respectively. Then*

$$\gamma(\pi\times\sigma,\psi)=q^{-\frac{m(n-m-1)}{2}}\omega_\sigma(-1)^{n-1}\epsilon_0(\pi\times\sigma,\psi).$$

3. Étale algebras and generalized Kloosterman sheaves

In this section, we review the definition of an étale algebra over $\mathbb{F}$. We specialize to the tensor product algebra $\mathbb{F}_n\otimes_{\mathbb{F}}\mathbb{F}_m$, and show that it can be used to rewrite the Gauss sums we encountered in the previous section in a more compact way. Finally, we give a brief review on the generalized Kloosterman sheaves constructed by Katz. We will use these sheaves later and relate them to values of the Bessel function.

3.1. Étale algebras.

Let B be a finite-dimensional $\mathbb{F}$-algebra. We say that B is an étale algebra of rank n if $B\cong\prod_{i=1}^r\mathbb{F}_{n_i}$, where $n_1,\dots,n_r$ are positive integers with $n_1+\dots+n_r=n$. We have that $B^\times\cong\prod_{i=1}^r\mathbb{F}_{n_i}^\times$. We denote by $\widehat{B^\times}$ the group of complex characters $\chi\colon B^\times\to\mathbb{C}^\times$. Then $\widehat{B^\times}\cong\prod_{i=1}^r\widehat{\mathbb{F}_{n_i}^\times}$.

Let R be an $\mathbb{F}$-algebra. We consider the tensor product $B\otimes_{\mathbb{F}}R$. We have the norm map, $\mathrm{N}_2^{B\otimes_{\mathbb{F}}R}\colon B\otimes_{\mathbb{F}}R\to R$, defined as follows. Let $x\in B\otimes_{\mathbb{F}}R$. We consider the multiplication map $T_x\colon B\otimes_{\mathbb{F}}R\to B\otimes_{\mathbb{F}}R$, $T_x(y)=xy$. The norm $\mathrm{N}_2^{B\otimes_{\mathbb{F}}R}(x)$ is defined as the determinant of T_x, viewed as a R-linear map . Similarly, we have the trace map $\mathrm{tr}_2^{B\otimes_{\mathbb{F}}R}\colon B\otimes_{\mathbb{F}}R\to R$ given by taking the trace of T_x, viewed as a R-linear map.

If R is a finite-dimensional $\mathbb{F}$-algebra, then we also have the corresponding norm $\mathrm{N}_1^{B\otimes_{\mathbb{F}}R}\colon B\otimes_{\mathbb{F}}R\to B$ and trace $\mathrm{tr}_1^{B\otimes_{\mathbb{F}}R}\colon B\otimes_{\mathbb{F}}R\to B$ maps to B, defined by taking the determinant and trace of T_x, this time viewed as a B-linear map. In this case, we also denote $\mathrm{tr}\,x=\mathrm{tr}\,T_x$, where we view T_x as an $\mathbb{F}$-linear map.

3.2. Tensor product of finite fields.

Let n,m be positive integers. Denote $d=\gcd(n,m)$, $l=\mathrm{lcm}(n,m)$. We consider the tensor product $\mathbb{F}_n\otimes_{\mathbb{F}}\mathbb{F}_m$.

We have that $\mathbb{F}_n\otimes_{\mathbb{F}}\mathbb{F}_m\cong\mathbb{F}_l^d$, by the following isomorphism. Write $\mathbb{F}_m=\mathbb{F}[\theta_m]=\mathbb{F}[X]/(p_m(X))$, where $p_m(X)\in\mathbb{F}[X]$ is an irreducible polynomial of degree m, and $\theta_m\in\mathbb{F}_m$ is a root of $p_m(X)$. Then $p_m(X)=\prod_{j=1}^m\left(X-\theta_m^{q^j}\right)$. We have $\mathbb{F}_n\otimes_{\mathbb{F}}\mathbb{F}_m\cong\mathbb{F}_n[X]/(p_m(X))$ and the last ring is isomorphic to $\mathbb{F}_l^d$ by mapping $P(X)\in\mathbb{F}_n[X]/(p_m(X))$ to $(P(\theta_m),P(\theta_m^{1/q}),\dots,P(\theta_m^{1/q^{d-1}}))$.

Under this isomorphism, $s_n\in\mathbb{F}_n$ acts on $(x_1,\dots,x_d)\in\mathbb{F}_l^d$ by $(s_n\otimes1)(x_1,\dots,x_d)=(s_nx_1,\dots,s_nx_d)$, while $s_m\in\mathbb{F}_m$ acts by $(1\otimes s_m)(x_1,\dots,x_d)=(s_mx_1,s_m^{1/q}x_2,\dots,s_m^{1/q^{d-1}}x_d)$.

We denote the norm maps defined in the previous section by $\mathrm{N}_1^{n,m}=\mathrm{N}_1^{\mathbb{F}_n\otimes_{\mathbb{F}}\mathbb{F}_m}\colon\mathbb{F}_n\otimes_{\mathbb{F}}\mathbb{F}_m\to\mathbb{F}_n$, $\mathrm{N}_2^{n,m}=\mathrm{N}_2^{\mathbb{F}_n\otimes_{\mathbb{F}}\mathbb{F}_m}\colon\mathbb{F}_n\otimes_{\mathbb{F}}\mathbb{F}_m\to\mathbb{F}_m$.

Under the above isomorphism, we have that the norms $N_1^{n,m}$, $N_2^{n,m}$ of $(x_1,\dots,x_d) \in \mathbb{F}_l^d$ are given by

$$N_1^{n,m}(x_1,\dots,x_d) = \prod_{j=1}^{d} \mathrm{N}_{l:n}(x_j),$$
$$N_2^{n,m}(x_1,\dots,x_d) = \prod_{j=1}^{d} \mathrm{N}_{l:m}(x_j)^{q^j}.$$

Similarly, under the above isomorphism, we have that the trace of $(x_1,\dots,x_d) \in \mathbb{F}_l^d$ to $\mathbb{F}$ is given by

$$\mathrm{tr}(x_1,\dots,x_d) = \sum_{j=1}^{d} \mathrm{Tr}_{\mathbb{F}_l/\mathbb{F}}(x_j).$$

This formalism allows us to rewrite the epsilon factors discussed previously in a cleaner way. Let $\alpha \in \widehat{\mathbb{F}_n^\times}$, $\beta \in \widehat{\mathbb{F}_m^\times}$. Then

$$\prod_{k=1}^{d} \tau\left(\alpha \circ \mathrm{N}_{l:n} \cdot \beta^{q^k} \circ \mathrm{N}_{l:m}, \psi_l\right) = (-1)^d \sum_{\xi \in (\mathbb{F}_n \otimes_{\mathbb{F}} \mathbb{F}_m)^\times} \alpha^{-1}\left(N_1^{n,m}(\xi)\right) \beta^{-1}\left(N_2^{n,m}(\xi)\right) \psi(\mathrm{tr}\,\xi). \tag{6}$$

We denote the value in eq. (6) by $\tau_{n,m}(\alpha,\beta,\psi)$. Since $d = \gcd(n,m)$ and $n+m+nm$ have the same parity, we have

$$\tau_{n,m}(\alpha,\beta,\psi) = (-1)^{nm+n+m} \sum_{\xi \in (\mathbb{F}_n \otimes_{\mathbb{F}} \mathbb{F}_m)^\times} \alpha^{-1}\left(N_1^{n,m}(\xi)\right) \beta^{-1}\left(N_2^{n,m}(\xi)\right) \psi(\mathrm{tr}\,\xi). \tag{7}$$

If $\lambda = (n_1,\dots,n_r) \vdash n$, we denote $\mathbb{F}_\lambda = \prod_{i=1}^{r} \mathbb{F}_{n_i}$. Then we have that $\mathbb{F}_\lambda \otimes_{\mathbb{F}} \mathbb{F}_m \cong \prod_{i=1}^{r} \mathbb{F}_{n_i} \otimes \mathbb{F}_m$, and that for $x = (x_1,\dots,x_r) \in \prod_{i=1}^{r} \mathbb{F}_{n_i} \otimes_{\mathbb{F}} \mathbb{F}_m$, the norm maps are given by

$$\mathrm{N}_1^{\mathbb{F}_\lambda \otimes_{\mathbb{F}} \mathbb{F}_m}(x) = \left(\mathrm{N}_1^{n_1,m}(x_1),\dots,\mathrm{N}_1^{n_r,m}(x_r)\right),$$
$$\mathrm{N}_2^{\mathbb{F}_\lambda \otimes_{\mathbb{F}} \mathbb{F}_m}(x) = \prod_{i=1}^{r} \mathrm{N}_2^{n_i,m}(x),$$

and the trace map (to $\mathbb{F}$) is given by

$$\mathrm{tr}\,x = \sum_{i=1}^{r} \mathrm{tr}\,x_i.$$

For $\alpha = (\alpha_1,\dots,\alpha_r) \in \widehat{\mathbb{F}_\lambda^\times} = \prod_{i=1}^{r} \widehat{\mathbb{F}_{n_i}^\times}$, $\beta \in \widehat{\mathbb{F}_m^\times}$, denote

$$\tau_{\lambda,m}(\alpha,\beta,\psi) = (-1)^{nm+n+rm} \sum_{\xi \in (\mathbb{F}_\lambda \otimes_{\mathbb{F}} \mathbb{F}_m)^\times} \alpha^{-1}(\mathrm{N}_1^{\mathbb{F}_\lambda \otimes_{\mathbb{F}} \mathbb{F}_m}(\xi)) \beta^{-1}(\mathrm{N}_2^{\mathbb{F}_\lambda \otimes_{\mathbb{F}} \mathbb{F}_m}(\xi)) \psi(\mathrm{tr}\,\xi).$$

Then we have that

$$\prod_{j=1}^{r} \tau_{n_i,m}(\alpha_i,\beta,\psi) = \tau_{\lambda,m}(\alpha,\beta,\psi).$$

3.3. **Katz's generalized Kloosterman sheaves.** Let B be a finite étale $\mathbb{F}$-algebra of rank n. We consider the following diagram of schemes over $\mathbb{F}$:

$$\begin{array}{ccc} & \operatorname{Res}_{B/\mathbb{F}} \mathbb{G}_m & \\ \swarrow^{\text{Norm}} & & \searrow^{\text{Trace}} \\ \mathbb{G}_m & & \mathbb{A}^1 \end{array},$$

where Norm, Trace are the norm and trace maps. For a commutative $\mathbb{F}$-algebra R, we have $\operatorname{Res}_{B/\mathbb{F}} \mathbb{G}_m (R) = (B \otimes_{\mathbb{F}} R)^{\times}$, $\operatorname{Norm}(R) = N_2^{B\otimes_{\mathbb{F}} R} \colon (B \otimes_{\mathbb{F}} R)^{\times} \to R^{\times}$, and $\operatorname{Trace}(R) = \operatorname{tr}_2^{B\otimes_{\mathbb{F}} R} \colon B \otimes_{\mathbb{F}} R \to R$.

Let ℓ be a prime different than the characteristic of $\mathbb{F}$. Let $\psi \colon B \to \overline{\mathbb{Q}_\ell}^{\times}$ be a non-degenerate additive character, and let $\chi \colon B^{\times} \to \overline{\mathbb{Q}_\ell}^{\times}$ be a multiplicative character. The additive character ψ gives an Artin-Schrier local system AS_ψ on $\mathbb{A}^1$. χ gives a Kummer local system on $\operatorname{Res}_{B/\mathbb{F}} \mathbb{G}_m$: the Lang cover $\pi \colon \operatorname{Res}_{B/\mathbb{F}} \mathbb{G}_m \to \operatorname{Res}_{B/\mathbb{F}} \mathbb{G}_m$ is a $\operatorname{Res}_{B/\mathbb{F}} \mathbb{G}_m (\mathbb{F}) = B^{\times}$-torsor, and $\mathcal{L}_\chi$ is the direct summand of $\pi_* \overline{\mathbb{Q}_\ell}$ on which $B^{\times}$ acts via χ. Consider the following ℓ-adic complex on $\mathbb{G}_m$:

$$\mathrm{Kl}(B, \chi, \psi) = \mathbf{R}\mathrm{Norm}_! \left(\mathrm{Trace}^* \mathrm{AS}_\psi \otimes \mathcal{L}_\chi\right)[n-1].$$

This complex has been introduced and studied by Katz in [14, Section 8.8]. See also [21, Appendix B]. It admits the following properties:

Theorem 3.1 ([14, Theorem 8.8.5])**.** *Denote $\mathcal{K} = \mathrm{Kl}(B, \chi, \psi)$. For $a \in \mathbb{F}^{\times}$, consider the trace of the geometric Frobenius at a acting on the stalk of $\mathcal{K}$. Then*

(1) $\mathcal{K}$ is a local system of rank n.
(2) $\mathcal{K}$ is pure of weight $n-1$. That is, the eigenvalues of $\mathrm{Fr}_a \mid \mathcal{K}_a$ are algebraic integers, and given an embedding $\overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$, the eigenvalues of $\mathrm{Fr}_a \mid \mathcal{K}_a$ and all of their algebraic conjugates have absolute value $q^{\frac{n-1}{2}}$.
(3) The trace of the m-th power of the geometric Frobenius at a acting on the stalk of $\mathcal{K}$ is given by

$$\operatorname{tr}\left(\mathrm{Fr}_a^m \mid \mathcal{K}_a\right) = (-1)^{n-1} \sum_{\substack{x \in B \otimes_{\mathbb{F}} \mathbb{F}_m \\ N_2^{B\otimes_{\mathbb{F}} \mathbb{F}_m}(x) = a}} \chi(N_1^{B\otimes_{\mathbb{F}} \mathbb{F}_m}(x)) \psi(\operatorname{tr} x).$$

If $B = \mathbb{F}_\lambda$, where $\lambda \vdash n$, we denote the sum appearing in Theorem 3.1 (3) by

$$J_{\lambda,m}(\chi, \psi, a) = \sum_{\substack{x \in \mathbb{F}_\lambda \otimes_{\mathbb{F}} \mathbb{F}_m \\ N_2^{\mathbb{F}_\lambda \otimes_{\mathbb{F}} \mathbb{F}_m}(x) = a}} \chi(N_1^{\mathbb{F}_\lambda \otimes_{\mathbb{F}} \mathbb{F}_m}(x)) \psi(\operatorname{tr} x),$$

and call $J_{\lambda,m}(\psi, \chi, a)$ a generalized unitary Kloosterman sum. These sums generalize the unitary Kloosterman sums introduced in [5].

4. A recursive expression for the Bessel function

In this section, we develop a recursive expression for the Bessel function. We begin with defining some parametrization for irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$ using characters compatible with a partition of n. This parametrization is defined even when the involved characters are not regular. Then we rewrite the recursive expression of the Bessel function (Theorem 2.14) using this parametrization. We relate this recursive expression to the generalized Kloosterman sheaves constructed by Katz. As a corollary, we are able to show that certain polynomials with Bessel function values coefficients have roots lying on the unit circle.

4.1. **Representations of a given type.** Let $n \geq 1$, and let $\lambda = (n_1, \dots, n_r)$ be a partition of n.

Given characters $\alpha_i \colon \mathbb{F}_{n_i}^\times \to \mathbb{C}^\times$ for every $1 \leq i \leq r$, we define an irreducible generic representation $\Pi_\lambda(\alpha_1, \dots, \alpha_r)$ as follows: let $f_1, \dots, f_l$ be all the Frobenius orbits of $\alpha_1, \dots, \alpha_r$ without repetitions. We define $\phi = \phi_\lambda(\alpha_1, \dots, \alpha_r) \in P_n(\Gamma)$ by $\phi(f_i) = (s_i)$, where

$$s_i = \frac{1}{d(f_i)} \sum_{\substack{1 \leq j \leq r \\ \alpha_j \in f_i}} n_j,$$

and $()$ outside of $f_1, \dots, f_l$. We denote $\Pi_\lambda(\alpha_1, \dots, \alpha_r) = \Pi_{\phi_\lambda(\alpha_1, \dots, \alpha_r)}$.

Proposition 4.1. *Let $\lambda = (n_1, \dots, n_r), \mu = (m_1, \dots, m_t)$ be partitions of n, m respectively, with $n > m$. Then*

$$\begin{aligned} &\epsilon_0(\Pi_\lambda(\alpha_1, \dots, \alpha_r) \times \Pi_\mu(\beta_1, \dots, \beta_t), \psi) = \\ &(-1)^{nm} q^{-\frac{nm}{2}} \prod_{i=1}^{r} \prod_{j=1}^{t} \prod_{k=1}^{\gcd(n_i, m_j)} \tau\left(\alpha_i \circ \mathrm{N}_{\mathrm{lcm}(n_i,m_j):n_i} \cdot \beta_j^{q^k} \circ \mathrm{N}_{\mathrm{lcm}(n_i,m_j):m_j}, \psi_{\mathrm{lcm}(n_i,m_j)}\right) \end{aligned} \tag{8}$$

for any choice of characters $\alpha_i \colon \mathbb{F}_{n_i}^\times \to \mathbb{C}^\times$, $\beta_j \colon \mathbb{F}_{m_j}^\times \to \mathbb{C}^\times$.

Proof. Let $d_1 = d(\alpha_1), \dots, d_r = d(\alpha_r)$, $u_1 = d(\beta_1), \dots, u_t = d(\beta_t)$ be the degrees of the Frobenius orbits generated by the characters. Let $\alpha_i' \in \widehat{\mathbb{F}_{d_i}^\times}$, $\beta_j' \in \widehat{\mathbb{F}_{u_j}^\times}$, such that $\alpha_i = \alpha_i' \circ \mathrm{N}_{n_i:d_i}$, $\beta_j = \beta_j' \circ \mathrm{N}_{m_j:u_j}$. Since $\alpha_i \circ \mathrm{N}_{\mathrm{lcm}(n_i,m_j):n_i} = \alpha_i' \circ \mathrm{N}_{\mathrm{lcm}(n_i,m_j):d_i} = \alpha_i' \circ \mathrm{N}_{\mathrm{lcm}(d_i,u_j):d_i} \circ \mathrm{N}_{\mathrm{lcm}(n_i,m_j):\mathrm{lcm}(d_i,u_j)}$, and similarly, $\beta_j^{q^k} \circ \mathrm{N}_{\mathrm{lcm}(n_i,m_j):m_j} = {\beta_j'}^{q^k} \circ \mathrm{N}_{\mathrm{lcm}(d_i,u_j):u_j} \circ \mathrm{N}_{\mathrm{lcm}(n_i,m_j):\mathrm{lcm}(d_i,u_j)}$, we have by the Hasse-Davenport relation that

$$\begin{aligned} &\tau\left(\alpha_i \circ \mathrm{N}_{\mathrm{lcm}(n_i,m_j):n_i} \cdot \beta_j^{q^k} \circ \mathrm{N}_{\mathrm{lcm}(n_i,m_j):m_j}, \psi_{\mathrm{lcm}(n_i,m_j)}\right) \\ =&\tau\left(\alpha_i' \circ \mathrm{N}_{\mathrm{lcm}(d_i,u_j):d_i} \cdot {\beta_j'}^{q^k} \circ \mathrm{N}_{\mathrm{lcm}(d_i,u_j):u_j}, \psi_{\mathrm{lcm}(d_i,u_j)}\right)^{\frac{\mathrm{lcm}(n_i,m_j)}{\mathrm{lcm}(d_i,u_j)}}. \end{aligned}$$

We notice that the product of the Gauss sums satisfies

$$\begin{aligned} &\prod_{k=1}^{\gcd(n_i,m_j)} \tau\left(\alpha_i \circ \mathrm{N}_{\mathrm{lcm}(n_i,m_j):n_i} \cdot \beta_j^{q^k} \circ \mathrm{N}_{\mathrm{lcm}(n_i,m_j):m_j}, \psi_{\mathrm{lcm}(n_i,m_j)}\right) \\ =&\left(\prod_{k=1}^{\gcd(d_i,u_j)} \tau\left(\alpha_i' \circ \mathrm{N}_{\mathrm{lcm}(d_i,u_j):d_i} \cdot {\beta_j'}^{q^k} \circ \mathrm{N}_{\mathrm{lcm}(d_i,u_j):u_j}, \psi_{\mathrm{lcm}(d_i,u_j)}\right)^{\frac{\mathrm{lcm}(n_i,m_j)}{\mathrm{lcm}(d_i,u_j)}}\right)^{\frac{\gcd(n_i,m_j)}{\gcd(d_i,u_j)}}, \end{aligned}$$

as the multiplicand repeats itself every $\gcd(d_i, u_j)$ terms (see Proposition A.1). Hence we have that

$$\prod_{k=1}^{\gcd(n_i,m_j)} \tau\left(\alpha_i \circ \mathrm{N}_{\mathrm{lcm}(n_i,m_j):n_i} \cdot \beta_j^{q^k} \circ \mathrm{N}_{\mathrm{lcm}(n_i,m_j):m_j}, \psi_{\mathrm{lcm}(n_i,m_j)}\right)$$
$$= \left(\prod_{k=1}^{\gcd(d_i,u_j)} \tau\left(\alpha_i' \circ \mathrm{N}_{\mathrm{lcm}(d_i,u_j):d_i} \cdot \beta_j'^{q^k} \circ \mathrm{N}_{\mathrm{lcm}(d_i,u_j):u_j}, \psi_{\mathrm{lcm}(d_i,u_j)}\right)\right)^{\frac{n_i m_j}{d_i u_j}}. \tag{9}$$

Denote by $f_1, \dots, f_l$ the different Frobenius orbits generated by $\alpha_1', \dots, \alpha_r'$, and by $g_1, \dots, g_s$ the different Frobenius orbits generated by $\beta_1', \dots, \beta_t'$. By eq. (9), Theorem 2.15, and using the fact that $nm = \sum_{i'=1}^{r} \sum_{j'=1}^{t} n_{i'} m_{j'}$, we can rewrite the right hand side of eq. (8) as

$$\prod_{i=1}^{l} \prod_{j=1}^{s} \prod_{\substack{1 \le i' \le r \\ 1 \le j' \le t \\ \alpha_{i'} \in f_i \\ \beta_{j'} \in g_j}} \epsilon_0\left(\Pi_{f_i} \times \Pi_{g_j}, \psi\right)^{\frac{n_{i'} m_{j'}}{d(f_i) d(g_j)}} = \prod_{i=1}^{l} \prod_{j=1}^{s} \epsilon_0\left(\Pi_{f_i} \times \Pi_{g_j}, \psi\right)^{\left(\sum_{\substack{1 \le i' \le r \\ \alpha_{i'} \in f_i}} \frac{n_{i'}}{d(f_i)}\right)\left(\sum_{\substack{1 \le j' \le t \\ \beta_{j'} \in g_j}} \frac{m_{j'}}{d(g_j)}\right)}.$$

By multiplicativity of the tensor product ϵ_0-factors (Theorem 2.16), and by the definitions of $\Pi_\lambda(\alpha_1, \dots, \alpha_r)$ and $\Pi_\mu(\beta_1, \dots, \beta_t)$, this product equals $\epsilon_0\left(\Pi_\lambda(\alpha_1, \dots, \alpha_r) \times \Pi_\mu(\beta_1, \dots, \beta_t), \psi\right)$, as required. $\square$

Using the formalism of étale algebras, we can restate Proposition 4.1 as

Proposition 4.2. *Let $\lambda = (n_1, \dots, n_r), \mu = (m_1, \dots, m_t)$ be partitions of n, m respectively, with $n > m$. Then*

$$\epsilon_0\left(\Pi_\lambda(\alpha_1, \dots, \alpha_r) \times \Pi_\mu(\beta_1, \dots, \beta_t), \psi\right) = (-1)^{nm} q^{-\frac{nm}{2}} \prod_{j=1}^{t} \tau_{\lambda, m_j}(\alpha, \beta_j, \psi),$$

for any choice of characters $\alpha = (\alpha_1, \dots, \alpha_r) \in \widehat{\mathbb{F}_\lambda^\times} = \prod_{i=1}^{r} \widehat{\mathbb{F}_{n_i}^\times}$, $\beta = (\beta_1, \dots, \beta_t) \in \widehat{\mathbb{F}_\mu^\times} = \prod_{j=1}^{t} \widehat{\mathbb{F}_{m_j}^\times}$.

Denote for a partition $\mu = (m_1, \dots, m_t) \vdash m$, the polynomial $\varphi_\mu(t) = \prod_{j=1}^{t} (t^{m_j} - 1)$, and $Z_\mu = \prod_{k=1}^{\infty} k^{\mu(k)} \cdot \mu(k)!$, where $\mu(k)$ is the number of times k appears in the partition μ, i.e., $\mu(k) = \#\{1 \le j \le t \mid m_j = k\}$. Denote $\mathbb{F}_\mu^\times = \prod_{j=1}^{t} \mathbb{F}_{m_j}^\times$, $\widehat{\mathbb{F}_\mu^\times} = \prod_{j=1}^{t} \widehat{\mathbb{F}_{m_j}^\times}$.

Lemma 4.3. *Let σ be an irreducible generic representation of $\mathrm{GL}_m(\mathbb{F})$. Suppose that σ is parameterized by $\phi \in P_m(\Gamma)$, which is supported on the pairwise distinct Frobenius orbits $f_1, \dots, f_r$ of degrees $d_1 = d(f_1), \dots, d_r = d(f_r)$, and suppose that $\phi(f_i) = (s_i)$. Then*

(1) There exists a surjective map

$$\left\{(\mu, \beta) \mid \mu \vdash m, \beta \in \widehat{\mathbb{F}_\mu^\times} \mid \Pi_\mu(\beta) \cong \sigma\right\} \to \left\{((\nu_1, \dots, \nu_r), (\gamma_1, \dots, \gamma_r)) \mid \nu_i \vdash s_i, \gamma_i \in f_i^{\mathfrak{n}(\nu_i)}\right\},$$

where $\mathfrak{n}(\nu_i)$ is the length of the partition ν_i, and $f_i^{\mathfrak{n}(\nu_i)} = f_i \times \cdots \times f_i$ is the Cartesian product set, where f_i appears in the product $\mathfrak{n}(\nu_i)$ times.

(2) *Let* $((\nu_1,\dots,\nu_r),(\gamma_1,\dots,\gamma_r))$ *belong to the set on the right hand side. Suppose* $\nu_i = (s_{i,1},\dots,s_{i,l_i})$*. Denote* $\mu_i = (d_i s_{i,1},\dots,d_i s_{i,l_i})$*, and let* μ *be the concatenation of the partitions* $\mu_1,\dots,\mu_r$*. Then the inverse image of* $((\nu_1,\dots,\nu_r),(\gamma_1,\dots,\gamma_r))$ *is of size*

$$\frac{Z_\mu}{d_1^{\mathfrak{n}(\nu_1)} Z_{\nu_1}\dots d_r^{\mathfrak{n}(\nu_r)} Z_{\nu_r}}.$$

If (μ',β') *is in the inverse image of* $((\nu_1,\dots,\nu_r),(\gamma_1,\dots,\gamma_r))$ *then* $\mu'=\mu$ *and* $\varphi_{\mu'}(q) = \prod_{i=1}^r \varphi_{\nu_i}(q^{d_i})$*.*

Proof. Let $\mu = (m_1,\dots,m_t) \vdash m$, $\beta = (\beta_1,\dots,\beta_t) \in \widehat{\mathbb{F}_\mu^\times}$ with $\Pi_\mu(\beta) \cong \sigma$. For every i let $1 \le j_{i,1} < \dots < j_{i,l_i} \le t$ be such that $\beta_{j_{i,1}},\dots,\beta_{j_{i,l_i}} \in f_i$. Then by the definition of $\Pi_\mu(\beta)$ we must have

$$\sum_{k=1}^{l_i} \frac{m_{j_{i,k}}}{d_i} = s_i,$$

and therefore $\nu_i = \left(\frac{m_{j_{i,1}}}{d_i},\dots,\frac{m_{j_{i,l_i}}}{d_i}\right)$ is a partition of s_i. Denote $\gamma_i = \left(\beta_{j_{i,1}},\dots,\beta_{j_{i,l_i}}\right) \in f_i^{l_i} = f_i^{\mathfrak{n}(\nu_i)}$. Therefore we constructed a map as in the theorem. We show that is surjective: given $\nu_1 \vdash s_1,\dots,\nu_r \vdash s_r$ and $\gamma_i \in f_i^{\mathfrak{n}(\nu_i)}$, write $\nu_i = (s_{i,1},\dots,s_{i,l_i})$, $\gamma_i = (\gamma_{i,1},\dots,\gamma_{i,l_i})$ Denote $\mu_i = (d_i s_{i,1},\dots,d_i s_{i,l_i})$, and denote by μ the concatenation of $\mu_1,\dots,\mu_r$. Let $\beta_{i,j} \in \widehat{\mathbb{F}_{d_i}^\times}$ representing $\gamma_{i,j}$ and let $\beta_i = \left(\beta_{i,1}\circ \mathrm{N}_{d_i s_{i,1}:d_i},\dots,\beta_{i,l_i}\circ \mathrm{N}_{d_i s_{i,l_i}:d_i}\right)$. Let β be the concatenation of $\beta_1,\dots,\beta_r$. Then $\Pi_\mu(\beta) \cong \sigma$ by definition and it is mapped under our map to $((\nu_1,\dots,\nu_r),(\gamma_1,\dots,\gamma_r))$.

If (μ',β') is an inverse image of $((\nu_1,\dots,\nu_r),(\gamma_1,\dots,\gamma_r))$, then by our construction we must have that μ' is given by μ as constructed above. Denote $\mu = (m_1,\dots,m_t) \vdash m$. For every k let $\mu(k)$ be the number of times that k appears in μ, and for $1 \le i \le r$, let $u_{k,i} = 0$ if $d_i \nmid k$ and $u_{k,i} = \frac{k}{d_i}$, if $d_i \mid k$, and let $\nu_i(u_{k,i})$ be the number of times that $u_{k,i}$ appears in ν_i. Then we have $\binom{\mu(k)}{\nu_1(u_{k,1}),\dots,\nu_r(u_{k,r})}$ options for indices $\left(1 \le j_{i,1} < \dots < j_{i,\nu(u_{k,i})} \le r\right)_{i=1}^{\mathfrak{n}(\mu)}$ with $m_{j_{i,i'}} = u_{k,i}$. Therefore we have

$$\prod_{k=1}^{\infty} \binom{\mu(k)}{\nu_1(u_{k,1}),\dots,\nu_r(u_{k,r})} = \prod_{k=1}^{\infty} \frac{\mu(k)!}{\nu_1(u_{k,1})!\dots\nu_r(u_{k,r})!} = \prod_{k=1}^{\infty} \frac{\mu(k)!}{\nu_1(k)!\dots\nu_r(k)!}$$

options for choosing the indices for the sequences $\left(\beta_{i,i'}\circ \mathrm{N}_{d_i s_{i,i'}:d_i}\right)_{i'=1}^{l_i}$ in the preimage. This number equals

$$\prod_{k=1}^{\infty} \frac{(d_1 k)^{\nu_1(k)}\dots(d_r k)^{\nu_r(k)}\mu(k)!}{d_1^{\nu_1(k)}\dots d_r^{\nu_r(k)} k^{\nu_1(k)}\dots k^{\nu_r(k)}\nu_1(k)!\dots\nu_r(k)!},$$

which equals

$$\frac{Z_\mu}{d_1^{\mathfrak{n}(\nu_1)}\dots d_r^{\mathfrak{n}(\nu_r)} Z_{\nu_1}\dots Z_{\nu_r}},$$

since $\sum_{k=1}^{\infty}\nu_i(k) = \mathfrak{n}(\nu_i)$ and since by definition

$$\prod_{k=1}^{\infty} k^{\mu(k)} = \prod_{k=1}^{\infty} (d_1 k)^{\nu_1(k)}\dots(d_r k)^{\nu_r(k)}.$$

The identity $\varphi_\mu(q) = \prod_{i=1}^r \varphi_{\nu_i}\left(q^{d_i}\right)$ is immediate from the definitions of μ and $\varphi_\mu(t)$. □

Theorem 4.4. *Let π be an irreducible generic representation of $\mathrm{GL}_n(\mathbb{F})$. Then for any $m < n$ and $g \in \mathrm{GL}_m(\mathbb{F})$, we have*

$$\mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ g & \end{pmatrix} = \sum_{\mu \vdash m} \frac{1}{Z_\mu} \frac{1}{\varphi_\mu(q)} \sum_{\beta \in \widehat{\mathbb{F}_\mu^\times}} \gamma\left(\pi \times \Pi_\mu(\beta)^\vee, \psi\right) \cdot \mathcal{J}_{\Pi_\mu(\beta),\psi}(g).$$

Proof. By Theorem 2.14 It suffices to show that for every irreducible generic representation σ of $\mathrm{GL}_m(\mathbb{F})$, we have that

$$\frac{1}{[\mathrm{GL}_m(\mathbb{F}) : U_m]} \dim \sigma = \sum_{\mu \vdash m} \frac{1}{Z_\mu} \frac{1}{\varphi_\mu(q)} \cdot \#\left\{\beta \in \widehat{\mathbb{F}_\mu^\times} \mid \Pi_\mu(\beta) \cong \sigma\right\}.$$

Suppose that σ is parameterized by $\phi \in P_m(\Gamma)$, which is supported on the pairwise distinct Frobenius orbits $f_1, \dots, f_r$ of degrees $d_1 = d(f_1), \dots, d_r = d(f_r)$, and suppose that $\phi(f_i) = (s_i)$. In this case we have

$$\dim \sigma = [\mathrm{GL}_m(\mathbb{F}) : U_m] \prod_{i=1}^r \frac{q^{\frac{d_i s_i (s_i - 1)}{2}}}{\prod_{j=1}^{s_i} \left(q^{d_i j} - 1\right)}.$$

By Lemma 4.3, we have

$$\sum_{\mu \vdash m} \sum_{\substack{\beta \in \widehat{\mathbb{F}_\mu^\times} \\ \Pi_\mu(\beta) \cong \sigma}} \frac{1}{Z_\mu} \frac{1}{\varphi_\mu(q)} = \sum_{\nu_1 \vdash s_1, \dots, \nu_r \vdash s_r} \sum_{\gamma_1 \in f_1^{\mathfrak{n}(\nu_1)}, \dots, \gamma_r \in f_r^{\mathfrak{n}(\nu_r)}} \frac{1}{Z_\mu} \frac{1}{\varphi_\mu(q)} \frac{Z_\mu}{d_1^{\mathfrak{n}(\nu_1)} \dots d_r^{\mathfrak{n}(\nu_r)} Z_{\nu_1} \dots Z_{\nu_r}},$$

where on the right hand side $\mu = \mu(\nu_1, \dots, \nu_r)$ is defined in Lemma 4.3 and satisfies $\varphi_\mu(q) = \prod_{i=1}^r \varphi_{\nu_i}\left(q^{d_i}\right)$. Since $\# f_i^{\mathfrak{n}(\nu_i)} = d_i^{\mathfrak{n}(\nu_i)}$, we need to show that

$$\prod_{i=1}^r \frac{q^{\frac{d_i s_i (s_i - 1)}{2}}}{\prod_{j=1}^{s_i} \left(q^{d_i j} - 1\right)} = \sum_{\nu_1 \vdash s_1} \cdots \sum_{\nu_r \vdash s_r} \frac{1}{\prod_{i=1}^r Z_{\nu_i} \varphi_{\nu_i}\left(q^{d_i}\right)}.$$

By equation (2.14') [17, Page 25], applied to Example 5 on [17, Page 27], we have the following identity of rational functions:

$$\frac{q^{\frac{s(s-1)}{2}}}{\prod_{k=1}^s \left(q^k - 1\right)} = \sum_{\lambda \vdash s} \frac{1}{Z_\lambda \varphi_\lambda(q)},$$

which implies our identity, by substituting $q \mapsto q^{d_i}$, $s \mapsto s_i$ for every i, and multiplying these equalities. □

Combining with Theorem 2.18, we get the following formula:

Corollary 4.5. *Let π be an irreducible generic representation of $\mathrm{GL}_n(\mathbb{F})$. Then for any $m < n$ and $g \in \mathrm{GL}_m(\mathbb{F})$, we have*

$$\mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ g & \end{pmatrix} = q^{-\frac{m(n-m-1)}{2}} \sum_{\mu \vdash m} \frac{1}{Z_\mu} \frac{1}{\varphi_\mu(q)} \times \sum_{\beta \in \widehat{\mathbb{F}_\mu^\times}} \beta(-1)^{n-1} \epsilon_0\left(\pi \times \Pi_\mu(\beta)^\vee, \psi\right) \mathcal{J}_{\Pi_\mu(\beta),\psi}(g).$$

4.2. **Relation with generalized Kloosterman sheaves.** We now relate the result in Theorem 4.4 to Katz's generalized Kloosterman sheaves.

Suppose that $\pi = \Pi_\lambda(\alpha)$, where $\lambda = (n_1, \dots, n_r) \vdash n$ and $\alpha = (\alpha_1, \dots, \alpha_r) \in \widehat{\mathbb{F}_\lambda^\times} = \prod_{i=1}^r \widehat{\mathbb{F}_{n_i}^\times}$.

We fix a prime ℓ different from the characteristic of $\mathbb{F}$, and fix an embedding $\overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$. Using this embedding, the characters ψ, α give rise to characters $\psi \colon \mathbb{F} \to \overline{\mathbb{Q}_\ell}^\times$, $\alpha \colon \mathbb{F}_\lambda^\times \to \overline{\mathbb{Q}_\ell}^\times$.

Theorem 4.6. *Let $\mathcal{K} = \mathrm{Kl}(\mathbb{F}_\lambda, \alpha^{-1}, \psi)$. Then for any $c \in \mathbb{F}^\times$ and any $0 \le m \le n$*

$$\left((-1)^{r-1} q^{-\frac{(n-1)}{2}}\right)^m \mathrm{tr}\left(\wedge^m\left(\mathrm{Fr}_{(-1)^{n-1}c^{-1}} \mid \mathcal{K}_{(-1)^{n-1}c^{-1}}\right)\right) = q^{\frac{m(n-m)}{2}} \mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ cI_m & \end{pmatrix},$$

where $\wedge^m$ is the m-th exterior power, and $\mathrm{Fr}_{(-1)^{n-1}c^{-1}} \mid \mathcal{K}_{(-1)^{n-1}c^{-1}}$ is the action of the geometric Frobenius at $(-1)^{n-1}c^{-1}$ acting on the stalk of $\mathcal{K}$.

Remark 4.7.
(1) Recall that the eigenvalues of $\mathrm{Fr}_{(-1)^{n-1}c^{-1}} \mid \mathcal{K}_{(-1)^{n-1}c^{-1}}$ all have absolute value $q^{\frac{n-1}{2}}$, and therefore the eigenvalues of $\wedge^m\left(\mathrm{Fr}_{(-1)^{n-1}c^{-1}} \mid \mathcal{K}_{(-1)^{n-1}c^{-1}}\right)$ all have absolute value $q^{\frac{m(n-1)}{2}}$. Hence, the factor on the left hand side is just a normalization.
(2) $c^{-1}(-1)^{n-1}$ is the determinant of the inverse of the matrix $\left(\begin{smallmatrix} & I_{n-1} \\ c & \end{smallmatrix}\right)$.
(3) If $m = 1$, we get the formula

$$\mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-1} \\ c & \end{pmatrix} = (-1)^{n+r} q^{-n+1} \sum_{\substack{\xi \in \mathbb{F}_\lambda^\times \\ \prod_{i=1}^r \mathrm{N}_{n_i:1}(\xi) = (-1)^{n-1}c^{-1}}} \alpha^{-1}(\xi)\, \psi(\mathrm{tr}\, \xi).$$

This formula is known in the literature for cuspidal representations [28, 20] and for generic representations [6, Lemma 3.5].
(4) Formulas similar to the one given in Theorem 4.6 are known in the literature for several cases. For instance, a special case of the Shintani formula [25] gives the formula

$$\mathrm{tr}\left(\wedge^m(A_\pi)\right) = q^{\frac{m(n-m)}{2}} W^\circ\begin{pmatrix} \varpi I_m & \\ & I_{n-m} \end{pmatrix},$$

where W° is the normalized spherical Whittaker function of an unramified representation π of $\mathrm{GL}_n(F)$, with Satake parameter A_π. Here F is a local non-archimedean field with a uniformizer ϖ, and the Whittaker model is taken with respect to a character ψ with conductor 0.
(5) The Bessel function satisfies the relation

$$\overline{\mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ cI_m & \end{pmatrix}} = \omega_\pi(c)^{-1} \mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_m \\ cI_{n-m} & \end{pmatrix}$$

This identity reminds of the binomial identity

$$\binom{n}{m} = \binom{n}{n-m}.$$

Both of these identities could be understood as a corollary of the isomorphism

$$\left(\wedge^m(V)\right)^\vee \cong \wedge^{n-m}(V) \otimes (\det V)^{-1}, \qquad (10)$$

where V is an n-dimensional complex representation of a group G. For the identity of the Bessel function, take V to be the action of the Frobenius at $(-1)^{n-1} c^{-1}$ acting on the stalk of $\mathcal{K}$, twisted by the character that sends the Frobenius to $(-1)^{r-1} q^{-\frac{n-1}{2}}$, and apply Theorem 4.6 and use the isomorphism in eq. (10), and the fact that all eigenvalues of the Frobenius are of absolute value 1.

Proof. Choose $g = cI_m$ in Corollary 4.5, where $c \in \mathbb{F}^\times$. Then $\mathcal{J}_{\Pi_\mu(\beta),\psi}(cI_m) = \beta(c)$, and we have that

$$\mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ cI_m & \end{pmatrix} = q^{-\frac{m(n-m-1)}{2}} \sum_{\mu \vdash m} \frac{1}{Z_\mu} \frac{1}{\varphi_\mu(q)} \sum_{\beta \in \widehat{\mathbb{F}_\mu^\times}} \epsilon_0\left(\pi \times \Pi_\mu(\beta)^\vee, \psi\right) \beta\left((-1)^{n-1} c\right).$$

Using Proposition 4.2, we have that for $\mu = (m_1, \dots, m_t) \vdash m$,

$$\begin{aligned}&\sum_{\beta \in \widehat{\mathbb{F}_\mu^\times}} \epsilon_0\left(\Pi_\lambda(\alpha) \times \Pi_\mu(\beta)^\vee, \psi\right) \beta\left((-1)^{n-1} c\right) \\ &= (-1)^{nm} q^{-\frac{nm}{2}} \sum_{\beta_1 \in \widehat{\mathbb{F}_{m_1}^\times}} \cdots \sum_{\beta_t \in \widehat{\mathbb{F}_{m_t}^\times}} \prod_{j=1}^t \beta_j\left((-1)^{n-1} c\right) \tau_{\lambda, m_j}\left(\alpha, \beta_j^{-1}, \psi\right).\end{aligned}$$

Using the fact that the sum of a non-trivial character over a group is zero, and the fact that $\left|\widehat{\mathbb{F}_\mu^\times}\right| = \varphi_\mu(q)$, we get that

$$\begin{aligned}&\frac{1}{\varphi_\mu(q)} \sum_{\beta \in \widehat{\mathbb{F}_\mu^\times}} \epsilon_0\left(\Pi_\lambda(\alpha) \times \Pi_\mu(\beta)^\vee, \psi\right) \beta\left((-1)^{n-1} c\right) \\ &= (-1)^{nt+mr} q^{-\frac{nm}{2}} \prod_{j=1}^t J_{\lambda, m_j}\left(\psi, \alpha^{-1}, (-1)^{n-1} c^{-1}\right),\end{aligned}$$

where $J_{\lambda,m_j}\left(\psi, \alpha^{-1}, (-1)^{n-1} c^{-1}\right)$ are the unitary Kloosterman sums introduced in Section 3.3.

Therefore, we have that

$$\begin{aligned}&q^{\frac{(n-m)m}{2}} \mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ cI_m & \end{pmatrix} \\ &= \left((-1)^{r-1} q^{-\frac{n-1}{2}}\right)^m \sum_{\mu = (m_1, \dots, m_t) \vdash m} \frac{1}{Z_\mu} (-1)^{t+m} \prod_{j=1}^t (-1)^{n-1} J_{\lambda, m}\left(\psi, \alpha^{-1}, (-1)^{n-1} c^{-1}\right).\end{aligned}$$

The theorem now follows from Theorem 3.1 (3) and from the following identity, which is true for any square complex valued matrix A:

$$\operatorname{tr}\left(\wedge^m(A)\right) = \sum_{\mu = (m_1, \dots, m_t) \vdash m} \frac{1}{Z_\mu} (-1)^{m+t} \prod_{j=1}^t \operatorname{tr}\left(A^{m_j}\right).$$

∎

4.3. **Polynomials with unitary roots.** We now show some applications of the relation we established in Theorem 4.6 between values of the Bessel function and the generalized Kloosterman sheaves.

Let $\pi = \Pi_\lambda(\alpha)$ be an irreducible generic representation of $\mathrm{GL}_n(\mathbb{F})$, where $\lambda = (n_1, \dots, n_r) \vdash n$, $\alpha \in \widehat{\mathbb{F}_\lambda^\times}$, and let $c \in \mathbb{F}^\times$.

Theorem 4.8. *The following polynomials have roots lying on the unit circle:*

$$P(X) = \sum_{m=0}^{n} \mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_m \\ cI_{n-m} & \end{pmatrix} q^{\frac{m(n-m)}{2}} X^m,$$

$$Q(X) = \sum_{m=0}^{n} \mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_m \\ cI_{n-m} & \end{pmatrix} X^m.$$

Proof. Let $A = \mathrm{Fr}_{(-1)^{n-1}c^{-1}} \mid \mathcal{K}_{(-1)^{n-1}c^{-1}}$ as in Theorem 4.6. The characteristic polynomial of A is given by

$$\mathrm{Char}_A(X) = \sum_{m=0}^{n} (-1)^{n-m} \operatorname{tr}\left(\wedge^{n-m}(A)\right) X^m.$$

By Theorem 3.1, we have that the roots of $\mathrm{Char}_A(X)$ have absolute value $q^{\frac{n-1}{2}}$. By Theorem 4.6, we have that

$$\mathrm{Char}_A(X) = q^{\frac{n(n-1)}{2}} \sum_{m=0}^{n} \left((-1)^r q^{\frac{m}{2}}\right)^{n-m} \mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_m \\ cI_{n-m} & \end{pmatrix} \left(q^{-\frac{n-1}{2}} X\right)^m.$$

Hence, we have that $P(X) = q^{-\frac{n(n-1)}{2}} (-1)^{rn} \mathrm{Char}_A((-1)^r q^{\frac{n-1}{2}} X)$, so the roots of $P(X)$ are all of absolute value 1.

For the result for $Q(X)$, we use Proposition B.1 by noticing that $Q(X) = P_{q^{-\frac{1}{2}}}(X)$. ☐

Using the fact that we have a formula for the trace of powers of $\mathrm{Fr}_{(-1)^{n-1}c^{-1}} \mid \mathcal{K}_{(-1)^{n-1}c^{-1}}$ (Theorem 3.1 (3)), we get the following result, which can be seen as a generalization of the results of [5].

Theorem 4.9. *(1) Let*

$$C(X) = \sum_{m=0}^{n} (-1)^{r(n-m)} q^{\frac{(n-m)(n+m-1)}{2}} \mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_m \\ cI_{n-m} & \end{pmatrix} X^m.$$

Let $z_1, \dots, z_n$ be the roots of $C(X)$. Then for every m,

$$z_1^m + \dots + z_n^m = (-1)^{n-1} J_{\lambda,m}\left(\psi, \alpha^{-1}, (-1)^{n-1} c^{-1}\right).$$

(2) Let

$$C^*(X) = \sum_{m=0}^{n} (-1)^{rm} q^{\frac{m(2n-m-1)}{2}} \mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-m} \\ cI_m & \end{pmatrix} X^m.$$

Write

$$C^*(X) = \prod_{i=1}^{n} (1 - \omega_i X),$$

where $\omega_1, \dots, \omega_n \in \mathbb{C}$. Then for every m,

$$\omega_1^m + \dots + \omega_n^m = (-1)^{n-1} J_{\lambda,m}\left(\psi, \alpha^{-1}, (-1)^{n-1} c^{-1}\right).$$

Proof. In order to prove (1), we notice that by Theorem 4.6, $C(X)$ is the characteristic polynomial of $\mathrm{Fr}_{(-1)^{n-1}c^{-1}} \mid \mathcal{K}_{(-1)^{n-1}c^{-1}}$, and that $\sum_{i=1}^{n} z_i^m$ is the trace of the m-th power of $\mathrm{Fr}_{(-1)^{n-1}c^{-1}} \mid \mathcal{K}_{(-1)^{n-1}c^{-1}}$. Now we use Theorem 3.1 (3).

In order to prove (2), we notice that $C^*(X) = X^n C(X^{-1})$, and therefore, without loss of generality, $\omega_i = z_i$. Now the result follows from (1). $\square$

Appendix A. Lemmas about Gauss sums

In this appendix, we give several lemmas about Gauss sums.

Proposition A.1. *Let $\alpha \in \widehat{\mathbb{F}_n^\times}$, $\beta \in \widehat{\mathbb{F}_m^\times}$. Let $d = \gcd(n,m)$, $l = \operatorname{lcm}(n,m)$. Then for any i,*

$$\tau\left(\alpha \circ \mathrm{N}_{l:n} \cdot \beta^{q^{i+d}} \circ \mathrm{N}_{l:m}, \psi_l\right) = \tau\left(\alpha \circ \mathrm{N}_{l:n} \cdot \beta^{q^{i}} \circ \mathrm{N}_{l:m}, \psi_l\right).$$

Proof. Recall that Gauss sums are constant on Frobenius orbits. Therefore, for any $r \in \mathbb{Z}$ we have

$$\tau\left(\alpha \circ \mathrm{N}_{l:n} \cdot \beta^{q^{i}} \circ \mathrm{N}_{l:m}, \psi_l\right) = \tau\left(\alpha \circ \mathrm{N}_{l:n}^{q^r} \cdot \beta^{q^{i}} \circ \mathrm{N}_{l:m}^{q^r}, \psi_l\right).$$

There exist $a, b \in \mathbb{Z}$, such that $an + bm = d$. Then

$$\begin{aligned}\tau\left(\alpha \circ \mathrm{N}_{l:n} \cdot \beta^{q^{i+d}} \circ \mathrm{N}_{l:m}, \psi_l\right) &= \tau\left(\alpha \circ \mathrm{N}_{l:n} \cdot \beta^{q^{i}} \circ \mathrm{N}_{l:m}^{q^d}, \psi_l\right)\\ &= \tau\left(\alpha \circ \mathrm{N}_{l:n} \cdot \beta^{q^{i}} \circ \mathrm{N}_{l:m}^{q^{an+bm}}, \psi_l\right).\end{aligned}$$

Since the image of $\mathrm{N}_{l:m}$ is in $\mathbb{F}_m^\times$, we have that $\mathrm{N}_{l:m}^{q^{bm}} = \mathrm{N}_{l:m}$. Similarly, the image of $\mathrm{N}_{l:n}$ is in $\mathbb{F}_n^\times$, and therefore $\mathrm{N}_{l:n}^{q^{an}} = \mathrm{N}_{l:n}$. Hence we get

$$\begin{aligned}\tau\left(\alpha \circ \mathrm{N}_{l:n} \cdot \beta^{q^{i+d}} \circ \mathrm{N}_{l:m}, \psi_l\right) &= \tau\left(\alpha \circ \mathrm{N}_{l:n}^{q^{an}} \cdot \beta^{q^{i}} \circ \mathrm{N}_{l:m}^{q^{an}}, \psi_l\right)\\ &= \tau\left(\alpha \circ \mathrm{N}_{l:n} \cdot \beta^{q^{i}} \circ \mathrm{N}_{l:m}, \psi_l\right),\end{aligned}$$

$\square$

Next, we prove a property about Gauss sums of $\mathbb{F}_{2m}^\times$. This result is needed for proving that the factors $\epsilon_0(\pi \times \sigma, \psi)$ and $\gamma(\pi \times \sigma, \psi)$ agree for $\pi \cong \sigma^\vee$ in Theorem 2.17.

Proposition A.2. *Let $\beta \in \widehat{\mathbb{F}_{2m}^\times}$ be a non-trivial character $\beta\colon \mathbb{F}_{2m}^\times \to \mathbb{C}^\times$, such that β is trivial on $\mathbb{F}_m^\times$. Then*

$$\tau(\beta, \psi_{2m}) = -q^m \beta^{-1}(z),$$

where $z \in \mathbb{F}_{2m}^\times$ satisfies $z^{q^m-1} = -1$.

Proof. Write

$$\tau(\beta, \psi_{2m}) = -\sum_{x \in \mathbb{F}_{2m}^\times} \beta^{-1}(x)\,\psi_{2m}(x).$$

Since β is trivial on $\mathbb{F}_m^\times$, we may write

$$-\tau(\beta, \psi_{2m}) = \frac{1}{q^m - 1} \sum_{a \in \mathbb{F}_m^\times} \sum_{x \in \mathbb{F}_{2m}^\times} \beta^{-1}\left(a^{-1}x\right) \psi_m\left(\mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}(x)\right).$$

Changing variable $x = ay$, we obtain

$$-\tau\left(\beta, \psi_{2m}\right) = \frac{1}{q^m - 1} \sum_{a \in \mathbb{F}_m^\times} \sum_{y \in \mathbb{F}_{2m}^\times} \beta^{-1}\left(y\right) \psi_m\left(a \mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}\left(y\right)\right).$$

We divide the sum into two sums according to whether the trace is zero or not:

$$\begin{aligned} -\tau\left(\beta, \psi_{2m}\right) = &\frac{1}{q^m - 1} \sum_{\substack{y \in \mathbb{F}_{2m}^\times \\ \mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}(y) \neq 0}} \sum_{a \in \mathbb{F}_m^\times} \beta^{-1}\left(y\right) \psi_m\left(a \mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}\left(y\right)\right) \\ &+ \frac{1}{q^m - 1} \sum_{\substack{y \in \mathbb{F}_{2m}^\times \\ \mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}(y) = 0}} \sum_{a \in \mathbb{F}_m^\times} \beta^{-1}\left(y\right) \psi_m\left(a \mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}\left(y\right)\right) \end{aligned}$$

Regarding first sum, since $\mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}\left(y\right)$ is non-zero, we may change variables and get

$$\sum_{a \in \mathbb{F}_m^\times} \psi_m\left(a \mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}\left(y\right)\right) = -1.$$

Hence we have,

$$\begin{aligned} -\tau\left(\beta, \psi_{2m}\right) &= -\frac{1}{q^m - 1} \sum_{\substack{y \in \mathbb{F}_{2m}^\times \\ \mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}(y) \neq 0}} \beta^{-1}\left(y\right) + \frac{1}{q^m - 1} \sum_{\substack{y \in \mathbb{F}_{2m}^\times \\ \mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}(y) = 0}} \sum_{a \in \mathbb{F}_m^\times} \beta^{-1}\left(y\right) \\ &= -\frac{1}{q^m - 1} \sum_{\substack{y \in \mathbb{F}_{2m}^\times \\ \mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}(y) \neq 0}} \beta^{-1}\left(y\right) + \sum_{\substack{y \in \mathbb{F}_{2m}^\times \\ \mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}(y) = 0}} \beta^{-1}\left(y\right). \end{aligned}$$

Since β is a non-trivial character, we have that $\sum_{y \in \mathbb{F}_{2m}^\times} \beta^{-1}\left(y\right) = 0$, and therefore we can write

$$-\tau\left(\beta, \psi_{2m}\right) = \frac{q^m}{q^m - 1} \sum_{\substack{y \in \mathbb{F}_{2m}^\times \\ \mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}(y) = 0}} \beta^{-1}\left(y\right).$$

Since the trace map $\mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m} \colon \mathbb{F}_{2m} \to \mathbb{F}_m$ is not injective, there exists $0 \neq z \in \mathbb{F}_{2m}$, such that $\mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}\left(z\right) = z + z^{q^m} = 0$. If $y \in \mathbb{F}_{2m}$ satisfies $\mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}\left(y\right) = 0$, i.e., $y^{q^m} + y = 0$, then $y^{q^m - 1} = -1$, and then

$$\left(\frac{y}{z}\right)^{q^m - 1} = 1,$$

i.e., $\frac{y}{z} \in \mathbb{F}_m^\times$. Therefore $y = c \cdot z$, where $c \in \mathbb{F}_m^\times$. Conversely, if $c \in \mathbb{F}_m^\times$, then $\mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}\left(cz\right) = c\mathrm{Tr}_{\mathbb{F}_{2m}/\mathbb{F}_m}\left(z\right) = 0$. Therefore, we have

$$-\tau\left(\beta, \psi_{2m}\right) = \frac{q^m}{q^m - 1} \sum_{c \in \mathbb{F}_m^\times} \beta^{-1}\left(cz\right) = q^m \beta^{-1}\left(z\right),$$

where in the last step we used again the fact that β is trivial on $\mathbb{F}_m^\times$. □

Appendix B. Polynomials with unitary roots

In this appendix, we prove a property about polynomials with roots lying on the unit circle. We were not able to find a reference for this property in the literature.

The result we are interested in is the following:

Proposition B.1. *Let*

$$P(X) = \sum_{k=0}^{n} a_k X^k \in \mathbb{C}[X]$$

be a polynomial, such that all of its roots lie on the unit circle. Then for any $-1 < \delta < 1$*, the polynomial*

$$P_\delta(X) = \sum_{k=0}^{n} a_k \delta^{k(n-k)} X^k$$

also has all of its roots lying on the unit circle.

We first recall the Lee-Yang theorem [15] [1, Theorem 8.4]. For a positive integer n, let $[n] = \{1, \dots, n\}$. For a complex number z, denote by $\overline{z}$ its complex conjugate.

Theorem B.2 (Lee-Yang)**.** *Let* $A = (a_{ij})_{i,j} \in \mathrm{Mat}_n(\mathbb{C})$ *be a hermitian matrix of size* $n \times n$*, i.e,* $a_{ij} = \overline{a_{ji}}$*. Suppose that* $|a_{ij}| \le 1$ *for every* i, j*. Then the following polynomial has all of its roots lying on the unit circle:*

$$\mathrm{LY}_A(X) = \sum_{S \subseteq [n]} \left(\prod_{i \in S, j \notin S} a_{ij} \right) X^{|S|},$$

where the empty product is defined to be 1*.*

We first show that certain polynomials satisfying the assumptions of Proposition B.1 can be represented as polynomials occurring in the Lee-Yang theorem.

Lemma B.3. *Suppose that*

$$P(X) = \sum_{k=0}^{n} a_k X^k$$

is a polynomial all of whose roots lie on the unit circle, such that $a_0 = a_n = 1$*. Then there exists a hermitian matrix* $A = (a_{ij}) \in \mathrm{Mat}_n(\mathbb{C})$*, such that* $|a_{ij}| = 1$ *for every* i, j*, with* $P(X) = \mathrm{LY}_A(X)$*.*

Proof. Write

$$P(X) = \prod_{i=1}^{n} (X + z_i),$$

where $z_i \in \mathbb{C}$ with $|z_i| = 1$ and $\prod_{i=1}^{n} z_i = 1$. Consider the following hermitian matrix $A = (a_{ij}) \in \mathrm{Mat}_n(\mathbb{C})$ with $|a_{ij}| = 1$ for every i, j:

$$a_{ij} = \begin{cases} 1 & i, j \ne n \text{ or } i = j = n, \\ z_i & i \ne n, j = n, \\ \overline{z_j} & i = n, j \ne n \end{cases}.$$

We claim that $\mathrm{LY}_A(X) = P(X)$. Let $S \subseteq [n]$. Note that $\prod_{i\in S, j\notin S} a_{ij} = 1$ if $S = [n]$ or if $S = \emptyset$, so we may assume $\emptyset \subsetneq S \subsetneq [n]$. We compute $\prod_{i\in S, j\notin S} a_{ij}$. There are two cases to consider: $n \in S$ and $n \notin S$.

If $n \notin S$, then

$$\prod_{i\in S, j\notin S} a_{ij} = \prod_{i\in S} a_{in} = \prod_{i\in S} z_i.$$

If $n \in S$, then

$$\prod_{i\in S, j\notin S} a_{ij} = \prod_{j\notin S} a_{nj} = \prod_{j\notin S} \overline{z_j} = \prod_{j\notin S} z_j^{-1} = \prod_{j\in S} z_j,$$

where we used the facts that $z_j\overline{z_j} = 1$ and that $\prod_{j=1}^{n} z_j = 1$.

Therefore, in all cases, we obtain

$$\prod_{i\in S, j\notin S} a_{ij} = \prod_{j\in S} z_j.$$

Therefore,

$$\mathrm{LY}_A(X) = \sum_{S\subseteq[n]} \left(\prod_{j\in S} z_j\right) X^{|S|} = \prod_{i=1}^{n} (X + z_i) = P(X),$$

as required. □

We are now ready to prove Proposition B.1.

Proof. Let

$$P(X) = \sum_{k=0}^{n} a_k X^k$$

be a polynomial all of whose roots are of absolute value 1, and let $-1 < \delta < 1$. Without loss of generality, we may assume $a_n = 1$, as for $c \in \mathbb{C}$, $cP_\delta(X) = (cP)_\delta(X)$. Since all of the roots of $P(X)$ are of absolute value 1, we have that $|a_0| = 1$. Let $\zeta \in \mathbb{C}$ be such that $\zeta^n = a_0$. Then $|\zeta| = 1$ and $Q(X) = \zeta^{-n} P(\zeta X)$ is a monic polynomial of degree n all of whose roots are of absolute value 1, and its constant coefficient is 1. By Lemma B.3, we have that there exists a hermitian matrix $A = (a_{ij})$, with $|a_{ij}| = 1$, such that $Q(X) = \mathrm{LY}_A(X)$. We have that δA is still a hermitian matrix all of which entries are of absolute value not greater than 1. We have $Q_\delta(X) = \mathrm{LY}_{\delta A}(X)$. By the Lee-Yang theorem, the roots of $Q_\delta(X)$ are all of absolute value 1. Since $P_\delta(X) = \zeta^n Q_\delta(\zeta^{-1}X)$, and since ζ is of absolute value 1, we have that the roots of $P_\delta(X)$ are all of absolute value 1. □

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