ON MATRIX COEFFICIENTS INTEGRALS ASSOCIATED TO BESSEL MODELS: THE PRINCIPAL SERIES CASE

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Abstract. We define a local ingredient of the Ichino-Ikeda conjecture for local representations that are not tempered. The representations in consideration are representations parabolically induced from characters and a tempered representation.

1. Introduction

In the 90s, Gross and Prasad stated a fascinating conjecture relating the vanishing of a period associated to automorphic representations of two special orthogonal groups, to the value of the tensor product L-function of these representations at the point $s = \frac{1}{2}$. Later, in 2009, Ichino and Ikeda stated a beautiful refinement of this conjecture [12]. Their refinement expresses the period as product of an L-function and an infinite product of local periods. Later, in his PhD thesis, Harris stated the analogous conjecture for unitary groups [11]. Since then much progress has been done on the Ichino-Ikeda conjecture for unitary groups, see [28, 5, 4].

In the statement of the Ichino-Ikeda conjectures mentioned above, one assumes that the representations involved are irreducible cuspidal automorphic representations, which are tempered at all places. The temperedness assumption is crucial, as it allows one to define the local periods, which are a key ingredient for the statement of the conjecture. The Ramanujan conjecture speculates that irreducible cuspidal automorphic representations that lie in a generic packet are already tempered everywhere, see [20]. However, the Ramanujan conjecture does not seem at reach anywhere in the near future: not much progress has been done since [21].

In this work, we explain how to define the local periods for non-tempered representations. We focus on principal series representations. The starting point for our work is a result of Moeglin and Waldspurger which explains how can one define the local integral of matrix coefficients using "meromorphic continuation". We show that the quotient of this meromorphic continuation with the remaining factor of the local invariant is well defined in our domain of interest: this domain is guaranteed by the trivial bound for the Ramanujan conjecture for cuspidal automorphic representations.

1.1. **The Ichino-Ikeda conjecture.**

Note: This is the only section where we consider the global setting. In other sections, we will always consider the local setting.

Let F be a number field, and let E/F be a quadratic étale algebra. Let σ be the unique non-trivial involution of E/F as an F-algebra. Let $(V_n, \langle\cdot,\cdot\rangle)$ be a hermitian space of rank n over E, and let $V_{n+1} = V_n \oplus Ee$, where $\langle e, e\rangle \neq 0$. We consider the unitary groups $\mathrm{U}(V_n)$, $\mathrm{U}(V_{n+1})$ of V_n, V_{n+1}, respectively.

Let π_n, π_{n+1} be irreducible automorphic cuspidal representations of $\mathrm{U}(V_n)$, $\mathrm{U}(V_{n+1})$, respectively. The global Gan-Gross-Prasad conjecture [7] considers the period $(\varphi_n \in \pi_n,$

$\varphi_{n+1} \in \pi_{n+1}$)

$$\mathcal{P}_{\mathrm{GGP}}(\varphi_n, \varphi_{n+1}) = \int_{\mathrm{U}(V_n)(F)\backslash \mathrm{U}(V_n)(\mathbb{A}_F)} \varphi_n(g_n)\varphi_{n+1}(g_n) \mathrm{d}g_n,$$

and relates it to the special L-function value $L(\frac{1}{2}, \mathrm{BC}(\pi_n) \times \mathrm{BC}(\pi_{n+1}))$. In particular, it speculates that if the L-function value is zero, then the period $\mathcal{P}_{\mathrm{GGP}}$ is identically zero.

The Ichino-Ikeda conjecture can be seen as a refinement of this conjecture. It roughly states that the period can be written as an infinite product of local periods. To explain the motivation for this conjecture we explain the phenomenon in a special case.

Suppose that $E = F \times F$, then $\mathrm{U}(V_n) \cong \mathrm{GL}_n$, $\mathrm{U}(V_{n+1}) \cong \mathrm{GL}_{n+1}$. In this case, if π_n, π_{n+1} are irreducible automorphic cuspidal representations of GL_n, GL_{n+1} respectively, then π_n, π_{n+1} are generic, and it is known [19, Theorem 1.4] that

$$\mathcal{P}_{\mathrm{GGP}}(\varphi_n, \varphi_{n+1}) = L(\tfrac{1}{2}, \pi_{n+1} \times \pi_n) \prod_v \frac{\Psi_v(s, W_{n,v}, W_{n+1,v})}{L(s, \pi_{n+1,v} \times \pi_{n,v})} \mid_{s=\frac{1}{2}} .$$

Here, we realize $\pi_{n,v}$ (respectively $\pi_{n+1,v}$) via its Whittaker model with respect to an additive character $\psi : F \to \mathbb{C}^\times$ (respectively, with respect to $\psi^{-1} : F \to \mathbb{C}^\times$). φ_n, φ_{n+1} correspond to $\bigotimes_v W_{n,v}$, $\bigotimes_v W_{n+1,v}$, respectively, and $\Psi_v(s, W_{n,v}, W_{n+1,v})$ are the Rankin-Selberg integrals

$$\Psi_v(s, W_{n,v}, W_{n+1,v}) = \int_{N_n(F_v)\backslash \mathrm{GL}_n(F_v)} W_{n,v}(g_{n,v}) W_{n+1,v}(g_{n,v}) \left|\det g_{n,v}\right|_v^{s-\frac{1}{2}} \mathrm{d}g_{n,v}.$$

$\Psi_v(s, W_{n,v}, W_{n+1,v})$ converges for $\mathrm{Re}\, s$ large, and admits a meromorphic continuation to the entire plane, which we continue to denote by the same symbol. The quotient $\frac{\Psi_v(s, W_{n,v}, W_{n+1,v})}{L(s, \pi_{n+1,v} \times \pi_{n,v})}$ is an entire function, hence we can evaluate it at $s = \frac{1}{2}$.

The Ichino-Ikeda conjecture suggests that a formula of the following form should hold ($\varphi_n \in \pi_n$, $\varphi_{n+1} \in \pi_{n+1}$, $\varphi_n^\vee \in \pi_n^\vee, \varphi_{n+1}^\vee \in \pi_{n+1}^\vee$):

$$\mathcal{P}_{\mathrm{GGP}}(\varphi_n, \varphi_{n+1}) \mathcal{P}_{\mathrm{GGP}}(\varphi_n^\vee, \varphi_{n+1}^\vee) \sim L(\tfrac{1}{2}, \pi_{n+1}, \pi_n) \prod_v \mathcal{P}_{\pi_{n,v}, \pi_{n+1,v}}(\varphi_{n,v}, \varphi_{n+1,v}; \varphi_{n,v}^\vee, \varphi_{n+1,v}^\vee),$$

where $\varphi_n, \varphi_{n+1}, \varphi_n^\vee, \varphi_{n+1}^\vee$ correspond to $\bigotimes_v \varphi_{n,v}$, $\bigotimes_v \varphi_{n+1,v}$, $\bigotimes_v \varphi_{n,v}^\vee$, $\bigotimes_v \varphi_{n+1,v}^\vee$, respectively, and $\sim$ means that both sides are identical up to a well understood invertible rational number (independent of $\varphi_n, \varphi_{n+1}, \varphi_n^\vee, \varphi_{n+1}^\vee$). In order for this conjecture to make sense, we need to define $L(\frac{1}{2}, \pi_{n+1}, \pi_n)$ and $\mathcal{P}_{\pi_{n,v}, \pi_{n+1,v}}(\varphi_{n,v}, \varphi_{n+1,v}; \varphi_{n,v}^\vee, \varphi_{n+1,v}^\vee)$. We will also have to add some further assumptions on π_n, π_{n+1}.

We have that the assignment

$$(\varphi_n, \varphi_{n+1}; \varphi_n^\vee, \varphi_{n+1}^\vee) \mapsto \mathcal{P}_{\mathrm{GGP}}(\varphi_n, \varphi_{n+1}) \mathcal{P}_{\mathrm{GGP}}(\varphi_n^\vee, \varphi_{n+1}^\vee)$$

defines an element of

$$\mathrm{Hom}_{\mathrm{U}(V_n)(\mathbb{A}_F)}(\pi_n \otimes \pi_{n+1}, 1) \boxtimes \mathrm{Hom}_{\mathrm{U}(V_n)(\mathbb{A}_F)}(\pi_n^\vee \otimes \pi_{n+1}^\vee, 1).$$

Hence, we should take $\mathcal{P}_{\pi_{n,v}, \pi_{n+1,v}}$ to be in the space

$$\mathrm{Hom}_{\mathrm{U}(V_n)(F_v)}(\pi_{n,v} \otimes \pi_{n+1,v}, 1) \boxtimes \mathrm{Hom}_{\mathrm{U}(V_n)(F_v)}(\pi_{n,v}^\vee \otimes \pi_{n+1,v}^\vee, 1).$$

By [1, 2, 26], the latter space is of dimension at most 1. A distinguished element of the latter space is given, at least formally, by

$$\begin{aligned}&\alpha_{\pi_{n,v},\pi_{n+1,v}}(\varphi_{n,v},\varphi_{n+1,v};\varphi_{n,v}^\vee,\varphi_{n+1,v}^\vee)\\&=\int_{\mathrm{U}(V_n)(F)}\langle\pi_{n,v}(g_{n,v})\varphi_{n,v},\varphi_{n,v}^\vee\rangle\langle\pi_{n+1,v}(g_{n,v})\varphi_{n+1,v},\varphi_{n+1,v}^\vee\rangle\mathrm{d}g_{n,v}.\end{aligned}$$

The integral defining $\alpha_{\pi_{n,v},\pi_{n+1,v}}$ absolutely converges if $\pi_{n,v}$, $\pi_{n+1,v}$ are tempered. If $\pi_{n,v}$, $\pi_{n+1,v}$ are tempered and v is an unramified plane (with other certain assumptions), and $\varphi_{n,v}^\circ\varphi_{n+1,v}^\circ,\varphi_{n,v}^{\circ\vee},\varphi_{n+1,v}^{\circ\vee}$ are spherical vectors such that $\langle\varphi_{n,v}^\circ,\varphi_{n,v}^{\circ\vee}\rangle=\langle\varphi_{n+1,v}^\circ,\varphi_{n+1,v}^{\circ\vee}\rangle=1$, then

$$\alpha_{\pi_{n,v},\pi_{n+1,v}}(\varphi_{n,v}^\circ,\varphi_{n+1,v}^\circ;\varphi_{n,v}^{\circ\vee},\varphi_{n+1,v}^{\circ\vee})=\Delta_{n+1,v}\frac{L_{E_v}(\frac{1}{2},\mathrm{BC}(\pi_{n,v})\times\mathrm{BC}(\pi_{n+1,v}))}{L_{F_v}(1,\pi_{n,v},\mathrm{Ad})L_{F_v}(1,\pi_{n+1,v},\mathrm{Ad})},$$

where

$$\Delta_{n+1,v}=\prod_{i=1}^{n+1}L(i,\chi_{E_v/F_v}).$$

Here, $\chi_{E_v/F_v}:F_v^\times\to\mathbb{C}^\times$ is the trivial character if $E_v=F_v\times F_v$, and if E_v/F_v is a quadratic extension, then χ_{E_v/F_v} is the quadratic character associated to it.

The current Ichino-Ikeda conjecture for unitary groups [11] states that if $\pi_{n,v}$, $\pi_{n+1,v}$ are tempered for every v, then there exists a choice of Haar measures (independent of π_n, π_{n+1}), such that

$$\mathcal{P}_{\mathrm{GGP}}(\varphi_n,\varphi_{n+1})\mathcal{P}_{\mathrm{GGP}}(\varphi_n^\vee,\varphi_{n+1}^\vee)=\frac{1}{2^\beta}L(\tfrac{1}{2},\pi_{n+1},\pi_n)\prod_v\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}(\varphi_{n,v},\varphi_{n+1,v};\varphi_{n,v}^\vee,\varphi_{n+1,v}^\vee),$$

where $\beta\geq 0$ is some integer, and

$$\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}(\varphi_{n,v},\varphi_{n+1,v};\varphi_{n,v}^\vee,\varphi_{n+1,v}^\vee)=\frac{\alpha_{\pi_{n,v},\pi_{n+1,v}}(\varphi_{n,v},\varphi_{n+1,v};\varphi_{n,v}^\vee,\varphi_{n+1,v}^\vee)}{L(\frac{1}{2},\pi_{n+1,v},\pi_{n,v})},$$

where

$$L(s,\pi_{n+1,v},\pi_{n,v})=\Delta_{n+1,v}\frac{L_{E_v}(s,\mathrm{BC}(\pi_{n,v})\times\mathrm{BC}(\pi_{n+1,v}))}{L_{F_v}(s+\frac{1}{2},\pi_{n,v},\mathrm{Ad})L_{F_v}(s+\frac{1}{2},\pi_{n+1,v},\mathrm{Ad})}$$

and $L(\frac{1}{2},\pi_{n+1},\pi_n)$ is given by considering the meromorphic continuation of the Euler product

$$L(s,\pi_{n+1},\pi_n)=\prod_v L(s,\pi_{n+1,v},\pi_{n,v}).$$

We note that the current conjecture can be also stated without having to define $L(s,\pi_{n+1,v},\pi_{n,v})$ at every place: let S be a finite set of places such that for $v\notin S$, we have that v is unramified and that $\varphi_{n,v},\varphi_{n+1,v},\varphi_{n,v}^\vee,\varphi_{n+1,v}^\vee$ are spherical vectors as above. Then the Ichino-Ikeda conjecture can be formulated as

$$\mathcal{P}_{\mathrm{GGP}}(\varphi_n,\varphi_{n+1})\mathcal{P}_{\mathrm{GGP}}(\varphi_n^\vee,\varphi_{n+1}^\vee)=\frac{1}{2^\beta}L^S(\tfrac{1}{2},\pi_{n+1},\pi_n)\prod_{v\in S}\alpha_{\pi_{n,v},\pi_{n+1,v}}(\varphi_{n,v},\varphi_{n+1,v};\varphi_{n,v}^\vee,\varphi_{n+1,v}^\vee),$$

where $L^S(s,\pi_{n+1},\pi_n)$ is the partial Euler product

$$L^S(s,\pi_{n+1},\pi_n)=\prod_{v\notin S}L(s,\pi_{n+1,v},\pi_{n,v}).$$

The assumption that $\pi_{n,v}$, $\pi_{n+1,v}$ are tempered for every v is crucial for the statement of the Ichino-Ikeda conjecture, since otherwise the local periods $\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}$ are not defined. The generalized Ramanujan conjecture states that if π_n (respectively π_{n+1}) lies in a generic packet, then $\pi_{n,v}$ (respectively $\pi_{n+1,v}$) is already tempered for every v. However, this conjecture is far from being known.

We would like to bypass the Ramanujan conjecture, and state an Ichino-Ikeda conjecture, given that π_n, π_{n+1} lie in generic packets (equivalently, given that their base change is an isobaric sum of hermitian cuspidal representations). In order to do that, we need to define $L(s, \pi_{n+1}, \pi_n)$ and $\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}$ for every v. The definition of $L(s, \pi_{n+1}, \pi_n)$ is available thanks to the existence of weak base change [23, 15]. Hence, we need to define $\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}$. This is the objective of author's PhD thesis project. In this work, we provide a definition of $\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}$ for unramified places v.

We remark that in [22, Lemme 1.7], Moeglin and Waldspurger provided a meromorphic continuation for $\alpha_{\pi_{n,v},\pi_{n+1,v}}$, which is holomorphic under the assumption that the exponents $(\sigma_{\pi_n}(v,i))_i$, $(\sigma_{\pi_{n+1}}(v,i))_i$ of π_n, π_{n+1}, respectively, satisfy the inequalities

$$\max_i |\sigma_{\pi_n}(v,i)| < \frac{1}{2}, \quad \max_j \left|\sigma_{\pi_{n+1}}(v,j)\right| < \frac{1}{2}, \tag{1.1}$$

$$\max_{i,j} \left|\sigma_{\pi_n}(v,i) \pm \sigma_{\pi_{n+1}}(v,j)\right| < \frac{1}{2}. \tag{1.2}$$

While the inequalities in eq. (1.1) are known to be true (the trivial bound of Jacquet-Shalika [13, Corollary 2.5]), the inequality in eq. (1.2) is not known to be true. Our work shows that for unramified $\pi_{n,v}, \pi_{n+1,v}$, a holomorphic extension is possible for $\mathcal{P}_{\pi_{n,v},\pi_{n+1,v}}$, i.e., for the normalized version of $\alpha_{\pi_{n,v},\pi_{n+1,v}}$, under the assumption that the inequalities in eq. (1.1) hold.

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2. Preliminaries

2.1. **Unitary groups notations.** Let F be a p-adic field. Let q be the cardinality of the residue field of F. Let E/F be a quadratic étale algebra, that is E/F is a quadratic field extension or $E = F \times F$. Let σ be the unique non-trivial involution of E/F as an F-algebra.

Let $E^\times$ be the multiplicative group of E, i.e., the group consisting of all invertible elements of E. If E/F is a field extension, then $E^\times = E \setminus \{0\}$. Otherwise, if $E = F \times F$, then $E^\times = F^\times \times F^\times$, where $F^\times = F \setminus \{0\}$. In both cases, each character of $E^\times$ can be represented as a product of a unitary character and an unramified character. We will often say "Let $\chi \left|\cdot\right|^s$ be a character of $E^\times$", "s is imaginary", "$\operatorname{Re} s$ is large". By this we mean:

- If E/F is a field extension, then $s \in \mathbb{C}$, $\chi : E^\times \to \mathbb{C}^\times$ is a unitary character, and $\chi \left|\cdot\right|^s$ is the character $E^\times \to \mathbb{C}^\times$ given by $x \mapsto \chi(x) \cdot |x|_E^s$. We say *that s is imaginary* if $s \in \sqrt{-1} \cdot \mathbb{R}$. We say that $\operatorname{Re} s$ *is large* whenever the real part of s is large.
- If $E = F \times F$, then $s = (s_1, s_2) \in \mathbb{C}^2$, $\chi = (\chi_1, \chi_2)$, where $\chi_1, \chi_2 : F^\times \to \mathbb{C}^\times$ are unitary characters, and $\chi \left|\cdot\right|^s$ is the character $F^\times \times F^\times \to \mathbb{C}^\times$ given by $(x_1, x_2) \mapsto \chi_1(x_1)\chi_2(x_2) \cdot |x_1|_F^{s_1} |x_2|_F^{s_2}$. We say that s *is imaginary* if $s \in \sqrt{-1} \cdot \mathbb{R}^2$. We say that $\operatorname{Re} s$ *is large* whenever both real parts of s_1, s_2 are large.

Let $\omega_1 |\cdot|^{a_1}, \dots, \omega_r |\cdot|^{a_r}$ be characters of $E^\times$. Denote $\underline{a} = (a_r, \dots, a_1)$.

- If E/F is a field extension, we denote $\mathbb{C}[q^{\pm \underline{a}}] = \mathbb{C}[q^{a_i}, q^{-a_i}]_{i=1,\dots,r}$, $\operatorname{Re} \underline{a} = (\operatorname{Re} a_i)_{i=1}^r$, and $\|\operatorname{Re} \underline{a}\| = \max_{i=1,\dots,r} |\operatorname{Re} a_i|$.
- If $E = F \times F$, write $a_i = (a_{i1}, a_{i2}) \in \mathbb{C}^2$. We denote $\mathbb{C}[q^{\pm \underline{a}}] = \mathbb{C}[q^{a_i}, q^{-a_i}]_{i=1,\dots,r} = \mathbb{C}[q^{a_{ij}}, q^{-a_{ij}}]_{\substack{i=1,\dots,r \\ j=1,2}}$, $\operatorname{Re} \underline{a} = (\operatorname{Re} a_{ij})_{\substack{i=1,\dots,r \\ j=1,2}}$, and $\|\operatorname{Re} \underline{a}\| = \max_{\substack{i=1,\dots,r \\ j=1,2}} |\operatorname{Re} a_{ij}|$.

In both cases we denote by $\mathbb{C}(q^{\pm \underline{a}})$ the ring of fractions of $\mathbb{C}[q^{\pm \underline{a}}]$, which we refer to by *the ring of rational functions in* $q^{-\underline{a}}$.

Suppose that $(\mathbf{V}, \langle \cdot, \cdot \rangle)$ is a non-degenerate hermitian space of finite rank over E, with respect to the involution σ. Denote by $\mathrm{U}(\mathbf{V})$ the unitary group of $\mathbf{V}$, that is, the subgroup of $\mathrm{GL}_E(\mathbf{V})$ consisting of elements $g \in \mathrm{GL}_E(\mathbf{V})$, such that $\langle gv, gw \rangle = \langle v, w \rangle$, for every $v, w \in \mathbf{V}$.

A subspace $\boldsymbol{X} \subset \mathbf{V}$ is called *totally isotropic* if for every $x_1, x_2 \in \boldsymbol{X}$ we have $\langle x_1, x_2 \rangle = 0$. If $\boldsymbol{X}, \boldsymbol{Y} \subset \mathbf{V}$ are both totally isotropic subspaces, then we say that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are dual if the map $\boldsymbol{X} \times \boldsymbol{Y} \to E$, $(x, y) \mapsto \langle x, y \rangle$ is non-degenerate.

If $E = F \times F$, then $\mathbf{V}$ can be written as $\mathbf{V} = \boldsymbol{X}_{\mathbf{V}} \times \boldsymbol{X}_{\mathbf{V}}^\vee$, where $\boldsymbol{X}_{\mathbf{V}}$ is a vector space over F, and $\boldsymbol{X}_{\mathbf{V}}^\vee$ is its dual, and the hermitian product is given by $\langle (x, x^\vee), (y, y^\vee) \rangle = (\langle x, y^\vee \rangle, \langle y, x^\vee \rangle)$. In this case, the unitary group $\mathrm{U}(\mathbf{V})$ is isomorphic to $\mathrm{GL}_F(\boldsymbol{X}_{\mathbf{V}})$, by the isomorphism sending $g \in \mathrm{GL}_F(\boldsymbol{X}_{\mathbf{V}})$ to the map $\boldsymbol{X}_{\mathbf{V}} \times \boldsymbol{X}_{\mathbf{V}}^\vee \to \boldsymbol{X}_{\mathbf{V}} \times \boldsymbol{X}_{\mathbf{V}}^\vee$, defined by $(x, x^\vee) \mapsto (gx, x^\vee \circ g^{-1})$.

2.2. **Standard sections.** Let $(\mathbf{V}, \langle \cdot, \cdot \rangle)$ be a non-degenerate hermitian space. Suppose that we have a decomposition

$$\mathbf{V} = \boldsymbol{X}_r \oplus \cdots \oplus \boldsymbol{X}_1 \oplus \mathbf{V}' \oplus \boldsymbol{Y}_1 \oplus \cdots \oplus \boldsymbol{Y}_r,$$

where:

(1) $\mathbf{V}' \subset \mathbf{V}$ is a non-degenerate subspace.
(2) The spaces $\boldsymbol{X}_1, \dots, \boldsymbol{X}_r, \boldsymbol{Y}_1, \dots, \boldsymbol{Y}_r$ are totally isotropic and orthogonal to $\mathbf{V}$.
(3) For every $i \neq j$, $\boldsymbol{X}_i$ (respectively $\boldsymbol{Y}_i$) is orthogonal to $\boldsymbol{X}_j$, $\boldsymbol{Y}_j$.
(4) For every i, $\boldsymbol{X}_i$ and $\boldsymbol{Y}_i$ are dual.

Consider the following flag:

$$\mathcal{F} : \boldsymbol{X}_r \subset \boldsymbol{X}_r \oplus \boldsymbol{X}_{r-1} \subset \cdots \subset \boldsymbol{X}_r \oplus \cdots \oplus \boldsymbol{X}_1.$$

Let $P_{\mathcal{F}, \mathrm{U}(\mathbf{V})} \subset \mathrm{U}(\mathbf{V})$ be the parabolic subgroup stabilizing this flag. It has Levi part isomorphic to

$$\operatorname{Res}_{E/F} \mathrm{GL}_E(\boldsymbol{X}_r) \times \cdots \times \operatorname{Res}_{E/F} \mathrm{GL}_E(\boldsymbol{X}_1) \times \mathrm{U}(\mathbf{V}').$$

Let $\tau_1, \dots, \tau_r$ be admissible representations of the F-points of $\operatorname{Res}_{E/F} \mathrm{GL}_E(\boldsymbol{X}_1)$, $\dots$, $\operatorname{Res}_{E/F} \mathrm{GL}_E(\boldsymbol{X}_r)$ respectively, and let $\pi_{\mathbf{V}'}$ be an admissible representation of $\mathrm{U}(\mathbf{V}')$. Let $(1 |\cdot|^{a_i})_{i=1}^r$ be unramified characters of $E^\times$. We consider the (normalized) parabolic induction

$$\Pi_{\underline{a}} = \mathrm{I}_{P_{\mathcal{F}, \mathrm{U}(\mathbf{V})}}^{\mathrm{U}(\mathbf{V})} (|\det|^{a_r} \tau_r \boxtimes \cdots \boxtimes |\det|^{a_1} \tau_1 \boxtimes \pi_{\mathbf{V}'}).$$

Let $\mathcal{K} \subset \mathrm{U}(\mathbf{V})$ be a maximal compact subgroup in good position with respect to $P_{\mathcal{F}, \mathrm{U}(\mathbf{V})}$. We have the Iwasawa decomposition $\mathrm{U}(\mathbf{V}) = P_{\mathcal{F}, \mathrm{U}(\mathbf{V})} \mathcal{K}$. We say that a section $f^{\underline{a}} \in \Pi_{\underline{a}}$ is standard with respect to $\mathcal{K}$ if its restriction to the subgroup $\mathcal{K}$ is independent of $\underline{a}$, i.e., the value $f^{\underline{a}}(k)$ does not depend on $\underline{a}$ for any $k \in \mathcal{K}$. We say that a section $f^{\underline{a}} \in \Pi_{\underline{a}}$ is holomorphic (respectively meromorphic), if for any $g \in \mathrm{U}(\mathbf{V})$, there exist polynomials $(p_{g,i}(q^{-\underline{a}}))_{i=1}^{N_g} \subset$

$\mathbb{C}[q^{\pm \underline{a}}]$ (respectively rational functions $(p_{g,i}(q^{-\underline{a}}))_{i=1}^{N_g} \in \mathbb{C}(q^{\pm \underline{a}})$) and vectors $(v_{g,i})_{i=1}^{N_g} \subset \tau_r \boxtimes \cdots \boxtimes \tau_1 \boxtimes \pi_{\mathbf{V}'}$, such that $f^{\underline{a}}(g) = \sum_{i=1}^{N_g} p_{g,i}(q^{-\underline{a}}) v_{g,i}$. The subspace of holomorphic (respectively meromorphic) sections is invariant under the action of $\mathrm{U}(\mathbf{V})$.

2.3. **Doubling integrals.** In the 1980s, Piatetski-Shapiro and Rallis introduced an integral representation for the tensor product representation of representations of $G \times \mathrm{GL}_1$, where G is a classical group [24, 8]. Their construction relies only on matrix coefficients of G, and does not require the representation of G to have any model (such as a Whittaker model). This construction is now known as the doubling method. In this section, we give a brief overview on the doubling method. We refer the reader to [18] and [27] for standard references about the doubling method.

Let $(\mathbf{V}, \langle \cdot, \cdot \rangle)$ be a finite rank hermitian space over E. Let $\overline{\mathbf{V}}$ be a vector space isomorphic to $\mathbf{V}$, via an isomorphism $\mathbf{V} \to \overline{\mathbf{V}}$, $v \mapsto \overline{v}$, equipped with the hermitian product $\langle \overline{v}, \overline{w} \rangle = -\langle v, w \rangle$, for $v, w \in \mathbf{V}$. Let $\mathbf{V}^{\square} = \mathbf{V} \oplus \overline{\mathbf{V}}$, where we set $\mathbf{V}$ and $\overline{\mathbf{V}}$ to be orthogonal. Let $\mathbf{V}^{\Delta} = \{v + \overline{v} \mid v \in \mathbf{V}\}$. Then $\mathbf{V}^{\Delta}$ is a maximal totally isotropic subspace of $\mathbf{V}^{\square}$. Let $P_{\mathbf{V}^{\Delta}} \subset \mathrm{U}(\mathbf{V}^{\square})$ be the parabolic subgroup stabilizing the subspace $\mathbf{V}^{\Delta}$. Then $P_{\mathbf{V}^{\Delta}}$ has Levi part isomorphic to $\mathrm{GL}(\mathbf{V}^{\Delta})$. We have a character $\det_{\Delta} : P_{\mathbf{V}^{\Delta}} \to E^{\times}$ by projection to the Levi part $P_{\mathbf{V}^{\Delta}} \to \mathrm{GL}(\mathbf{V}^{\Delta})$ and composition with det.

We have an embedding $i : \mathrm{U}(\mathbf{V}) \times \mathrm{U}(\mathbf{V}) \to \mathrm{U}(\mathbf{V}^{\square})$ given by

$$i\left(g,h\right)\left(v\right) = gv,$$
$$i\left(g,h\right)\left(\overline{v}\right) = \overline{hv},$$

where $v \in \mathbf{V}$. We denote by $\Delta : \mathrm{U}(\mathbf{V}) \to \mathrm{U}(\mathbf{V}^{\square})$ the map $\Delta(g) = i(g,g)$. We actually have that the image of Δ is contained in the Levi part of $P_{\mathbf{V}^{\Delta}}$.

Let π be an admissible representation of $\mathrm{U}(\mathbf{V})$, $\chi \left|\cdot\right|^s$ be a character of $E^{\times}$. We consider the space $I(\chi \left|\cdot\right|^s, \mathbf{V}) = \mathrm{I}_{P_{\mathbf{V}^{\Delta}}}^{\mathrm{U}(\mathbf{V}^{\square})}((\chi \left|\cdot\right|^s) \circ \det_{\Delta})$. The doubling zeta integrals are defined via the formula

$$Z\left(f^s, v_{\pi}, v_{\pi}^{\vee}\right) = \int_{\Delta(\mathrm{U}(\mathbf{V}))\backslash \mathrm{U}(\mathbf{V}) \times \mathrm{U}(\mathbf{V})} f^s\left(i(g_1, g_2)\right) \langle \pi(g_1) v, \pi^{\vee}(g_2) v^{\vee} \rangle \chi^{-1}\left(\det g_2\right) \mathrm{d}(g_1, g_2),$$

where $f^s \in I(\chi \left|\cdot\right|^s, \mathbf{V})$ is a holomorphic section, $v_{\pi} \in \pi$, $v_{\pi}^{\vee} \in \pi^{\vee}$. These integrals are absolutely convergent for $\operatorname{Re} s$ large, which depends only on π. In this convergence domain, the integrals converge to holomorphic functions, which have a meromorphic continuation to the entire plane. We continue denoting the meromorphic continuation using the same notation.

We move to define L-factors of $\pi \times \chi$. If $E = F \times F$, then $\mathrm{U}(\mathbf{V})$ is a general linear group. In this case, we define for $\chi = (\chi_1, \chi_2)$, $s = (s_1, s_2)$,

$$L_{\mathrm{PSR}}(s + \tfrac{1}{2}, \pi \times \chi) = L_{\mathrm{GJ}}(s_1 + \tfrac{1}{2}, \pi \times \chi_1) L_{\mathrm{GJ}}(s_2 + \tfrac{1}{2}, \pi^{\vee} \times \chi_2),$$

where L_{GJ} are the L-factors of Godement-Jacquet. By [27, Lemma 5.3], for any holomorphic section $f^s \in I(\chi \left|\cdot\right|^s, \mathbf{V})$ and any $v_{\pi} \in \pi, v_{\pi}^{\vee} \in \pi^{\vee}$, the quotient

$$\frac{Z\left(f^s, v_{\pi}, v_{\pi}^{\vee}\right)}{L_{\mathrm{PSR}}(s + \frac{1}{2}, \pi \times \chi)}$$

is a polynomial (an element of $\mathbb{C}[q^{\pm s}]$).

If E/F is a quadratic extension, the definition goes through the greatest common divisor of a fractional ideal. There exists a notion of "good sections", see [18, 27, 16]. In particular, holomorphic sections are good. Consider the space

$$I_{\pi,\chi} = \operatorname{span}_{\mathbb{C}[q^{\pm s}]}\left\{Z\left(f^s, v_\pi, v_\pi^\vee\right) \mid f^s \in I(\chi\left|\cdot\right|^s, \mathbf{V}) \text{ is a good section}, v_\pi \in \pi, v_\pi^\vee \in \pi^\vee\right\}.$$

If π is irreducible, then there exists a "greatest common divisor" for $I_{\pi,\chi}$. To explain this, we first note that $I_{\pi,\chi}$ is a fractional ideal of $\mathbb{C}[q^{\pm s}]$ with $1 \in I_{\pi,\chi}$. There exists a unique polynomial $P(Z) \in \mathbb{C}[Z]$, such that $P(0) = 1$, and such that $I_{\pi,\chi} = \frac{1}{P(q^{-s})}\mathbb{C}[q^{\pm s}]$. We denote

$$L_{\mathrm{PSR}}(s + \tfrac{1}{2}, \pi \times \chi) = \frac{1}{P\left(q^{-s}\right)}.$$

When π and χ are unramified, we have that $L_{\mathrm{PSR}}(s, \pi \times \chi) = L(s, \mathrm{BC}(\pi) \times \chi)$ [27, Proposition 7.1].

2.4. **Rankin-Selberg integrals.** Suppose that $(V_n, \langle\cdot,\cdot\rangle)$ is a non-degenerate hermitian space of rank n over E. Let H be a hyperbolic plane. By this we mean a two-dimensional hermitian space over E with an orthogonal basis b_+, b_-, such that $\langle b_+, b_+\rangle = -\langle b_-, b_-\rangle \neq 0$. Let $V_{n+2} = V_n \oplus H$, where we set V_n and H to be orthogonal, and let $V_{n+1} = V_n \oplus Eb_+$.

Let $\mathrm{U}(V_{n+2})$ be the unitary group of V_{n+2}, and let $\mathrm{U}(V_{n+1})$ be the unitary group of V_{n+1}, realized as a subgroup of $\mathrm{U}(V_{n+2})$, consisting of all elements acting trivially on the vector b_-. Similarly, we realize $\mathrm{U}(V_n)$, the unitary group of V_n, as a subgroup of $\mathrm{U}(V_{n+1})$ consisting of all elements acting trivially on b_+.

Let $f_+ = b_+ + b_-$, $f_- = b_+ - b_-$. We have that f_+ and f_- are isotropic vectors, $\langle f_+, f_-\rangle = 2\langle b_+, b_+\rangle$, and that

$$V_{n+2} = Ef_+ \oplus V_n \oplus Ef_-.$$

Let Q be the parabolic subgroup of $\mathrm{U}(V_{n+2})$ stabilizing the line Ef_+. Then Q has Levi part isomorphic to $\operatorname{Res}_{E/F}\mathrm{GL}_E(Ef_+) \times \mathrm{U}(V_n) \cong \operatorname{Res}_{E/F} E^\times \times \mathrm{U}(V_n)$.

Let π_n, π_{n+1} be irreducible representations of $\mathrm{U}(V_n)$, $\mathrm{U}(V_{n+1})$ respectively. Let $\chi\left|\cdot\right|^s$ be a character of $E^\times \cong \mathrm{GL}_E(Ef_+)$. Let

$$\pi_{n+2,s} = \left|\cdot\right|^s \chi \times \pi_n = \mathrm{I}_Q^{\mathrm{U}(V_{n+2})}(\left|\cdot\right|^s \chi \boxtimes \pi_n).$$

Suppose $c_{\pi_n,\pi_{n+1}} \in \operatorname{Hom}_{\mathrm{U}(V_n)}(\pi_n \otimes \pi_{n+1}, 1)$. We consider the Rankin-Selberg integral

$$C_{\pi_{n+1},\pi_{n+2,s}}\left(v_{n+1}, f_{n+2}^s\right) = \int_{\mathrm{U}(V_n)\backslash \mathrm{U}(V_{n+1})} c_{\pi_n,\pi_{n+1}}(f_{n+2}^s(g_{n+1}), \pi_{n+1}(g_{n+1})v_{n+1})\mathrm{d}g_{n+1},$$

where $f_{n+2}^s \in \pi_{n+2,s}$ is a holomorphic section, and $v_{n+1} \in \pi_{n+1}$. By [14, Lemma 4.1], this integral converges for $\operatorname{Re} s$ large, depending only on π_n, π_{n+1}, and has a meromorphic continuation to the entire plane, which is a rational function in q^{-s}. By its definition, we get that in its convergence domain $C_{\pi_{n+1},\pi_{n+2,s}} \in \operatorname{Hom}_{\mathrm{U}(V_{n+1})}(\pi_{n+1} \otimes \pi_{n+2,s}, 1)$, and by the uniqueness theorem, this holds for the meromorphic continuation of $C_{\pi_{n+1},\pi_{n+2,s}}$ as well.

2.5. **Special sections.** In this section, we review a construction of Ginzburg–Piatetski-Shapiro–Rallis for sections of π_{n+2} from Section 2.4. See [9, Chapter 1], [25, Section 3].

Let $(V_n, \langle\cdot,\cdot\rangle)$ be a non-degenerate hermitian space of rank n over E, and let $V_{n+1} = V_n \oplus Eb$, where $\langle b, b\rangle \neq 0$, and we set b to be orthogonal to V_n. We take in Section 2.3 $\mathbf{V} = V_{n+1}$. We realize V_{n+2} from Section 2.4, as a subspace of $V_{n+1}^\square$, where $b_+ = b$, $b_- = \overline{b}$, i.e., we realize $V_{n+2} = Eb \oplus V_n \oplus E\overline{b} \subset V_{n+1}^\square$.

We have an embedding $i : \mathrm{U}(V_{n+2}) \times \mathrm{U}(V_n) \to \mathrm{U}(V_{n+1}^{\square})$ given by

$$i\left(g_{n+2}, g_n\right)\left(v_{n+2}\right) = g_{n+2} v_{n+2},$$
$$i\left(g_{n+2}, g_n\right)\left(\overline{v_n}\right) = \overline{g_n v_n},$$

where $g_{n+2} \in \mathrm{U}(V_{n+2})$, $g_n \in \mathrm{U}(V_n)$, $v_{n+2} \in V_{n+2}$, $v_n \in V_n$. We also have the embedding from Section 2.3, $i : \mathrm{U}(V_{n+1}) \times \mathrm{U}(V_{n+1}) \to \mathrm{U}(V_{n+1}^{\square})$, given by

$$i(g_{n+1}, h_{n+1})(v_{n+1}) = g_{n+1} v_{n+1},$$
$$i(g_{n+1}, h_{n+1})(\overline{v_{n+1}}) = \overline{h_{n+1} v_{n+1}},$$

where $g_{n+1}, h_{n+1} \in \mathrm{U}(V_{n+1})$, $v_{n+1} \in V_{n+1}$.

Let π_n, π_{n+1}, $\chi \left|\cdot\right|^s$, $\pi_{n+2,s} = \chi \left|\cdot\right|^s \times \pi_n$ be as in Section 2.4. Let

$$\rho_{\chi,s} = I(\chi \left|\cdot\right|^s, V_{n+1}) = \mathrm{I}_{P_{V_{n+1}^{\Delta}}}^{\mathrm{U}(V_{n+1}^{\square})}((\chi \left|\cdot\right|^s) \circ \det{}_{\Delta}).$$

Given a holomorphic section $f_\rho^s \in \rho_{\chi,s}$, $v_n \in \pi_n$, we consider the kernel integral

$$\Lambda_{f_\rho^s, v_n}(g_{n+2}) = \int_{\mathrm{U}(V_n)} f_\rho^s(i(g_{n+2}, g_n)) \chi^{-1}(\det g_n) \pi_n(g_n) v_n \mathrm{d}g_n.$$

This integral converges for $\mathrm{Re}\, s$ large enough, depending only on π_n (see the discussion in [25, Section 3]). It has a meromorphic continuation to the entire plane, which we continue to denote using the same symbol. We have that $\Lambda_{f_\rho^s, v_n}$ is a meromorphic section that lies in the space of $\pi_{n+2,s}$.

Let $c_{\pi_n, \pi_{n+1}} \in \mathrm{Hom}_{\mathrm{U}(V_n)}(\pi_n \otimes \pi_{n+1}, 1)$, and let $C_{\pi_{n+1}, \pi_{n+2,s}} : \pi_{n+1} \otimes \pi_{n+2,s} \to \mathbb{C}$ be the Rankin-Selberg integral introduced in Section 2.4. Then we have the following identity [9, Lemma 1.1], [25, Lemma 4.1]:

(2.1)
$$C_{\pi_{n+1}, \pi_{n+2,s}}(v_{n+1}, \Lambda_{f_\rho^s, v_n}) = \int_{\mathrm{U}(V_{n+1})} f_\rho^s(i(g_{n+1}, \mathrm{id}_{V_{n+1}})) c_{\pi_n, \pi_{n+1}}(v_n, \pi_{n+1}(g_{n+1}) v_{n+1}) \mathrm{d}g_{n+1}.$$

Here, the right hand side converges for $\mathrm{Re}\, s$ large depending on π_n, π_{n+1} only, and has a meromorphic continuation to the entire plane, which is a rational function in q^{-s}. The identity is then understood as an equality of meromorphic functions.

2.6. **Integrals of matrix coefficients.** Let us be in the setup of Section 2.4. The local integral considered in the statement of the Ichino-Ikeda conjecture [12, 11] is defined by the formula

$$\alpha_{\pi_n, \pi_{n+1}}(v_n, v_{n+1}; v_n^\vee, v_{n+1}^\vee) = \int_{\mathrm{U}(V_n)} \langle \pi_n(g_n) v_n, v_n^\vee \rangle \langle \pi_{n+1}(g_n) v_{n+1}, v_{n+1}^\vee \rangle \mathrm{d}g_n,$$

where $v_n \in \pi_n$, $v_{n+1} \in \pi_{n+1}$, $v_n^\vee \in \pi_n^\vee$, $v_{n+1}^\vee \in \pi_{n+1}^\vee$. This integral absolutely converges whenever the representations π_n, π_{n+1} are tempered [12, Proposition 1.1], [11, Proposition 2.1].

We have the following unramified computation. If E/F is a field extension, suppose that E/F is unramified. Suppose that π_n, π_{n+1} are unramified representations, then for the data $v_n^\circ \in \pi_n$, $v_{n+1}^\circ \in \pi_{n+1}$, $v_n^{\vee\circ} \in \pi_n^\vee$, $v_{n+1}^{\vee\circ} \in \pi_{n+1}^\vee$, where all vectors are spherical, and $\langle v_n^\circ, v_n^{\vee\circ}\rangle = \langle v_{n+1}^\circ, v_{n+1}^{\vee\circ}\rangle = 1$, we have [11, Section 2.2.3]:

$$\alpha_{\pi_n,\pi_{n+1}}(v_n^\circ, v_{n+1}^\circ; v_n^{\vee\circ}, v_{n+1}^{\vee\circ}) = \Delta_{n+1} \cdot \frac{L_E(\frac{1}{2}, \mathrm{BC}(\pi_n) \times \mathrm{BC}(\pi_{n+1}))}{L_F(1, \pi_n, \mathrm{Ad}) \cdot L_F(1, \pi_{n+1}, \mathrm{Ad})},$$

where $\Delta_{n+1} = \prod_{j=1}^{n+1} L(j, \chi_{E/F}^j)$, where $\chi_{E/F}$ is the trivial character, if $E = F \times F$, and $\chi_{E/F}$ is the quadratic character associated to E/F by class field theory, if E/F is a field extension.

Suppose that π_n, π_{n+1} are tempered. Consider $\pi_{n+2,s} = \chi\, |\cdot|^s \times \pi_n$ as in Section 2.4. If s is imaginary, then $\pi_{n+2,s}$ is tempered, and there exists a choice of Haar measures such that the following identity holds [3, eq. (7.4.9)]:

$$\begin{aligned}(2.2)\qquad & \alpha_{\pi_{n+1},\pi_{n+2,s}}(v_{n+1}, f_{n+2}^s; v_{n+1}^\vee, f_{n+2}^{\vee s}) \\ &= \int_{(\mathrm{U}(V_n)\backslash \mathrm{U}(V_{n+1}))^2} \mathrm{d}(g_{n+1}, g'_{n+1}) \\ &\quad \times \alpha_{\pi_n,\pi_{n+1}}(f_{n+2}^s(g_{n+1}), \pi_{n+1}(g_{n+1})v_{n+1}; f_{n+2}^{\vee s}(g'_{n+1}), \pi_{n+1}^\vee(g'_{n+1})v_{n+1}^\vee),\end{aligned}$$

where $v_{n+1} \in \pi_{n+1}$, $f_{n+2}^s \in \pi_{n+2,s}$, $v_{n+1}^\vee \in \pi_{n+1}^\vee$, $f_{n+2}^{\vee s} \in \pi_{n+2,s}^\vee$. In this case, the integral in eq. (2.2) absolutely converges [3, Claim (7.4.10)].

3. The main result

3.1. The representations considered. Let $(V_n, \langle\cdot,\cdot\rangle)$ be a non-degenerate hermitian space of rank n over E, and let $V_{n+1} = V_n \oplus Ee_+$, where e_+ is orthogonal to V_n and $\langle e_+, e_+\rangle \neq 0$.

Suppose that there exists a decomposition

$$V_n = X_l \oplus V_m \oplus Y_l,$$

where V_m is a non-degenerate subspace of rank m, and X_l, Y_l are totally isotropic subspaces of rank l, dual to each other. Let

$$\mathcal{F}_l : 0 \subset Ef_l \subset Ef_l \oplus Ef_{l-1} \subset \cdots \subset Ef_l \oplus \cdots \oplus Ef_1 = X_l$$

be a complete flag in X_l. Let $P_{\mathcal{F}_l,\mathrm{U}(V_n)}$ be the parabolic subgroup of $\mathrm{U}(V_n)$ stabilizing this flag. It has Levi part isomorphic to

$$(\mathrm{Res}_{E/F}\, E^\times)^l \times \mathrm{U}(V_m).$$

Similarly, suppose that there exists a decomposition

$$V_{n+1} = X_{l'} \oplus V_{m'} \oplus Y_{l'},$$

where $|m - m'| = 1$, with either $l = l'$ and $V_m \subset V_{m'}$, or $l' = l + 1$ and $V_{m'} \subset V_m$. Here again $X_{l'}$, $Y_{l'}$ are totally isotropic subspaces of rank l', dual to each other, and $V_{m'} \subset V_{n+1}$ is a non-degenerate hermitian subspace of rank m'. Choose a complete flag in $X_{l'}$:

$$\mathcal{F}'_{l'} : 0 \subset Ef'_{l'} \subset Ef'_{l'} \oplus Ef'_{l'-1} \subset \cdots \subset Ef'_{l'} \oplus \cdots \oplus Ef'_1 = X_{l'},$$

and let $P_{\mathcal{F}'_{l'},\mathrm{U}(V_{n+1})}$ be the parabolic subgroup of $\mathrm{U}(V_{n+1})$ stabilizing the flag $\mathcal{F}'_{l'}$. It has Levi part isomorphic to

$$(\mathrm{Res}_{E/F}\, E^\times)^{l'} \times \mathrm{U}(V_{m'}).$$

Let $\pi_m, \pi_{m'}$ be irreducible tempered representations of $\mathrm{U}(V_m), \mathrm{U}(V_{m'})$, respectively. Let $(\omega_i\,|\cdot|^{a_i})_{i=1}^{l}$, $(\mu_i\,|\cdot|^{b_i})_{i=1}^{l'}$ be sequences of characters of $E^\times$. We denote $\underline{a} = (a_i)_{i=1}^{l}$, $\underline{b} = (b_i)_{i=1}^{l'}$. We define

$$\pi_{n,\underline{a}} = \mathrm{I}_{P_{\mathcal{F}_l,\mathrm{U}(V_n)}}^{\mathrm{U}(V_n)}(\omega_l\,|\cdot|^{a_l} \boxtimes \cdots \boxtimes \omega_1\,|\cdot|^{a_1} \boxtimes \pi_m),$$
$$\pi_{n+1,\underline{b}} = \mathrm{I}_{P_{\mathcal{F}'_{l'},\mathrm{U}(V_{n+1})}}^{\mathrm{U}(V_{n+1})}(\mu_{l'}\,|\cdot|^{b_{l'}} \boxtimes \cdots \boxtimes \mu_1\,|\cdot|^{b_1} \boxtimes \pi_{m'}).$$

We say that $\underline{a}$ (respectively $\underline{b}$) is imaginary if a_i is imaginary (respectively b_i is imaginary), for every $1 \le i \le l$ (respectively $1 \le i \le l'$).

We will denote sections of these representations by $f_n^{\underline{a}} \in \pi_{n,\underline{a}}$, $f_{n+1}^{\underline{b}} \in \pi_{n+1,\underline{b}}$, etc.

3.1.1. *Construction of V_{n+2} and realization of π_{n+2}.* For induction purposes, we explain how to construct a space V_{n+2} containing V_{n+1}, so that its unitary group $\mathrm{U}(V_{n+2})$ will serve for a representation $\pi_{n+2,(s,\underline{a})}$. The construction is similar to the one as discussed in Section 2.4. Let $V_{n+2} = V_{n+1} \oplus Ee_-$, where e_- is orthogonal to V_{n+1}, and $\langle e_-, e_-\rangle = -\langle e_+, e_+\rangle$. Denote $f_+ = e_+ + e_-$, $f_- = e_+ - e_-$. Consider the flag

$$\mathcal{F}_{l+1} : 0 \subset Ef_+ \subset Ef_+ \oplus Ef_l \subset \cdots \subset Ef_+ \oplus Ef_l \oplus \cdots \oplus Ef_1 = Ef_+ \oplus X_l.$$

Let $P_{\mathcal{F}_{l+1},\mathrm{U}(V_{n+2})}$ be the parabolic subgroup of $\mathrm{U}(V_{n+2})$ stabilizing the flag $\mathcal{F}_{l+1}$. It has Levi part isomorphic to $(\mathrm{Res}_{E/F}\,E^\times)^{l+1} \times \mathrm{U}(V_m)$. Let $\chi\,|\cdot|^s$ be a character of $E^\times$. We denote

$$\pi_{n+2,(s,\underline{a})} = \mathrm{I}_{P_{\mathcal{F}_{l+1},\mathrm{U}(V_{n+2})}}^{\mathrm{U}(V_{n+2})}(\chi\,|\cdot|^s \boxtimes \omega_l\,|\cdot|^{a_l} \boxtimes \cdots \boxtimes \omega_1\,|\cdot|^{a_1} \boxtimes \pi_m).$$

On the other hand, we have the decomposition

$$V_{n+2} = Ef_+ \oplus V_n \oplus Ef_-,$$

where the subspaces Ef_+, Ef_- are isotropic lines, dual to each other. Let $P_{Ef_+,\mathrm{U}(V_{n+2})}$ be the parabolic subgroup of $\mathrm{U}(V_{n+2})$, stabilizing the subspace Ef_+. It has Levi part isomorphic to $\mathrm{Res}_{E/F}(E^\times) \times \mathrm{U}(V_n)$.

We realize $\pi_{n+2,(s,\underline{a})}$ via the parabolic induction $\mathrm{I}_{P_{Ef_+,\mathrm{U}(V_{n+2})}}^{\mathrm{U}(V_{n+2})}(\chi\,|\cdot|^s \boxtimes \pi_{n,\underline{a}})$. This is done using transitivity of induction: we map a section $F_{n+2}^{(s,\underline{a})} \in \mathrm{I}_{P_{Ef_+,\mathrm{U}(V_{n+2})}}^{\mathrm{U}(V_{n+2})}(\chi\,|\cdot|^s \boxtimes \pi_{n,\underline{a}})$ to the section $f_{n+2}^{(s,\underline{a})} \in \pi_{n+2,(s,\underline{a})}$ defined by $f_{n+2}^{(s,\underline{a})}(g_{n+2}) = F_{n+2}^{(s,\underline{a})}(g_{n+2})(\mathrm{id}_{V_n})$. Similarly, we realize $\pi_{n+2,(t,\underline{a})}^\vee$ via the parabolic induction $\mathrm{I}_{P_{Ef_+,\mathrm{U}(V_{n+2})}}^{\mathrm{U}(V_{n+2})}(\chi^{-1}\,|\cdot|^{-t} \boxtimes \pi_{n,\underline{a}}^\vee)$.

3.2. **Extension of the definition of the local integral.**

3.2.1. *Statement of the problem.* The Ichino-Ikeda conjecture considers a normalized version of α, the integral of matrix coefficients from 2.6, normalized so that the unramified computation gives the value 1, i.e., it considers the functional $\mathcal{P}_{\pi_n,\pi_{n+1}}$

$$\mathcal{P}_{\pi_n,\pi_{n+1}}(v_n, v_{n+1}; v_n^\vee, v_{n+1}^\vee) = \Delta_{n+1}^{-1} \cdot \frac{L_F(1,\pi_n,\mathrm{Ad}) \cdot L_F(1,\pi_{n+1},\mathrm{Ad})}{L_E(\frac{1}{2},\mathrm{BC}(\pi_n) \times \mathrm{BC}(\pi_{n+1}))} \times \alpha_{\pi_n,\pi_{n+1}}(v_n, v_{n+1}; v_n^\vee, v_{n+1}^\vee).$$

Our goal is to understand how to make sense of $\mathcal{P}_{\pi_n,\pi_{n+1}}$ for non-tempered representations. We will focus on the case $\pi_n = \pi_{n,\underline{a}}$, $\pi_{n+1} = \pi_{n+1,\underline{b}}$, i.e., the representations considered in Section 3.1. Furthermore, we will assume that $\|\mathrm{Re}\,\underline{a}\|, \|\mathrm{Re}\,\underline{b}\| < \frac{1}{2}$.

Our starting point is a result of Moeglin and Waldspurger [22, Lemme 1.7]. Their result is stated only for representations of special orthogonal groups, but the proof works also for unitary groups [10, Lemma 4.1.11].

Proposition 3.1. *For any standard sections $f_n^{\underline{a}} \in \pi_{n,\underline{a}}$, $f_{n+1}^{\underline{b}} \in \pi_{n+1,\underline{b}}$, $f_n^{\vee\underline{a}} \in \pi_{n,\underline{a}}^\vee$, $f_{n+1}^{\vee\underline{b}} \in \pi_{n+1,\underline{b}}^\vee$, the integral $\alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}})$ absolutely converges in the domain*

$$\mathcal{D} = \left\{ (\underline{a},\underline{b}) \mid \left\|\mathrm{Re}\,\underline{a}\right\|, \left\|\mathrm{Re}\,\underline{b}\right\| < \frac{1}{2}, \max_{\substack{u_i \in \mathrm{Re}\,\underline{a} \\ t_j \in \mathrm{Re}\,\underline{b}}} |u_i \pm t_j| < \frac{1}{2} \right\}.$$

Furthermore, there exists a polynomial $D(q^{-\underline{a}}, q^{-\underline{b}}) \in \mathbb{C}[q^{\pm\underline{a}}, q^{\pm\underline{b}}]$ that does not vanish in the domain $\mathcal{D}$, such that for every standard sections $f_n^{\underline{a}} \in \pi_{n,\underline{a}}$, $f_{n+1}^{\underline{b}} \in \pi_{n+1,\underline{b}}$, $f_n^{\vee\underline{a}} \in \pi_{n,\underline{a}}^\vee$, $f_{n+1}^{\vee\underline{b}} \in \pi_{n+1,\underline{b}}^\vee$, there exists a polynomial $L_{f_n^{\underline{a}}, f_{n+1}^{\underline{b}}, f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}}(q^{-\underline{a}}, q^{-\underline{b}}) \in \mathbb{C}[q^{\pm\underline{a}}, q^{\pm\underline{b}}]$, such that for $(\underline{a},\underline{b}) \in \mathcal{D}$,

$$\alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}) = \frac{L_{f_n^{\underline{a}}, f_{n+1}^{\underline{b}}, f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}}(q^{-\underline{a}}, q^{-\underline{b}})}{D(q^{-\underline{a}}, q^{-\underline{b}})}.$$

In particular, the assignment $(\underline{a},\underline{b}) \mapsto \alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}})$ has a meromorphic continuation to the entire plane.

Proposition 3.1 already gives an extension of $\alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}$ for non-tempered representations. The problem is that this meromorphic continuation is only defined in the domain $\mathcal{D}$, which requires the extra condition

$$\max_{\substack{u_i \in \mathrm{Re}\,\underline{a} \\ t_j \in \mathrm{Re}\,\underline{b}}} |u_i \pm t_j| < \frac{1}{2}.$$

This condition is not guaranteed to be satisfied by representations arising as local components of cuspidal automorphic representations lying in a generic packet. Our goal then is to try to find a refined version of the denominator polynomial $D(q^{-\underline{a}}, q^{-\underline{b}})$, which will allow us to define the normalized value $\mathcal{P}_{\pi_n,\pi_{n+1}}(v_n, v_{n+1}; v_n^\vee, v_{n+1}^\vee)$ for all $\underline{a}, \underline{b}$ satisfying $\left\|\mathrm{Re}\,\underline{a}\right\|, \left\|\mathrm{Re}\,\underline{b}\right\| < \frac{1}{2}$.

In the region $\left\|\mathrm{Re}\,\underline{a}\right\|, \left\|\mathrm{Re}\,\underline{b}\right\| < \frac{1}{2}$, we have that the assignments $\underline{a} \mapsto L(1, \pi_{n,\underline{a}}, \mathrm{Ad})$, $\underline{b} \mapsto L(1, \pi_{n+1,\underline{b}}, \mathrm{Ad})$ are holomorphic. Therefore, it suffices to find a holomorphic extension for the assignment

$$(\underline{a},\underline{b}) \mapsto \frac{\alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}})}{L_E(\frac{1}{2}, \mathrm{BC}(\pi_{n,\underline{a}}) \times \mathrm{BC}(\pi_{n+1,\underline{b}}))},$$

for $\underline{a}, \underline{b}$ satisfying $\left\|\mathrm{Re}\,\underline{a}\right\|, \left\|\mathrm{Re}\,\underline{b}\right\| < \frac{1}{2}$.

3.2.2. *Intuition from the split case.* In this section, we give a formal (but not rigorous) identity for $\alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}$ in the split case. Let $\mathfrak{o}$ be the ring of integers of F. Fix a non-trivial additive character $\psi : F \to \mathbb{C}^\times$. We normalize the measures, so that the volume of $\mathfrak{o}$ is 1.

In this case, $E = F \times F$, and $\mathrm{U}(V_n) \cong \mathrm{GL}_n(F)$, $\mathrm{U}(V_{n+1}) \cong \mathrm{GL}_{n+1}(F)$. We have that for $\underline{a}$, $\underline{b}$ outside of a finite union of hyperplanes, $\pi_{n,\underline{a}}$ and $\pi_{n+1,\underline{b}}$ are irreducible and generic. We realize $\pi_{n,\underline{a}}$ and $\pi_{n+1,\underline{b}}$ via their Whittaker models $\mathcal{W}(\pi_{n,\underline{a}}, \psi)$, $\mathcal{W}(\pi_{n+1,\underline{b}}, \psi^{-1})$, with respect to the corresponding upper triangular unipotent subgroups. See the discussion in [6, Section 3.1].

Let $B_m \subset \mathrm{GL}_m(F)$ be the upper triangular Borel subgroup, $A_m \subset \mathrm{GL}_m(F)$ be the diagonal subgroup, $N_m \subset \mathrm{GL}_m(F)$ be the upper unipotent subgroup. Let $K_m = \mathrm{GL}_m(\mathfrak{o})$ be the standard maximal compact subgroup of $\mathrm{GL}_m(F)$. We normalize the measures, so that K_m has volume 1. For $a_m = (a_{mi})_{i=1}^m$, let

$$\delta_{B_m}(a_m) = \prod_{1 \le i < j \le m} \left| \frac{a_i}{a_j} \right|$$

be the modular character. For $a_{m-1} \in A_{m-1}$, we have $\delta_{B_m}(a_{m-1}) = |\det a_{m-1}| \, \delta_{B_{m-1}}(a_{m-1})$, where we realize $A_{m-1} \subset A_m$ via the embedding $a_{m-1} \mapsto \mathrm{diag}(a_{m-1}, 1)$. We have the Iwasawa decomposition: if $f : \mathrm{GL}_m(F) \to \mathbb{C}$ is integrable, then

$$\int_{\mathrm{GL}_m(F)} f(g_m) \mathrm{d}g_m = \int_{N_m} \int_{A_m} \int_{K_m} \delta_{B_m}^{-1}(a_m) f(n_m a_m k_m) \mathrm{d}k_m \mathrm{d}a_m \mathrm{d}n_m.$$

We will use a formula by Lapid and Mao. Let $R_m \subset \mathrm{GL}_m(F)$ be the mirabolic subgroup, consisting of matrices having $(0, \dots, 0, 1)$ as their last row. Let τ_m be an irreducible unitarizable generic representation of $\mathrm{GL}_m(F)$. We have a non-degenerate $\mathrm{GL}_m(F)$-invariant pairing of $\tau_m \times \tau_m^\vee \to \mathbb{C}$, given by the formula

$$[W, W^\vee] = \int_{N_m \backslash R_m} W(g_m) W^\vee(g_m) \mathrm{d}g_m.$$

If τ_m is unramified, and $W^\circ \in \mathcal{W}(\tau_m, \psi)$, $W^{\circ\vee} \in \mathcal{W}(\tau_m^\vee, \psi^{-1})$ are spherical vectors with $W^\circ(I_m) = W^{\vee\circ}(I_m) = 1$, then we have, by the Iwasawa decomposition

$$\begin{aligned} [W^\circ, W^{\circ\vee}] &= \int_{A_{m-1}} \delta_{B_{m-1}}^{-1}(a_{m-1}) W^\circ(a_{m-1}) W^{\circ\vee}(a_{m-1}) \mathrm{d}a_{m-1} \\ &= \int_{A_m} \delta_{B_m}^{-1}(a_m) W^\circ(a_m) W^{\circ\vee}(a_m) \Phi(a_m e_m) \left|\det a_m\right| \mathrm{d}a_m, \end{aligned}$$

where $\Phi : F^m \to \mathbb{C}$ is the characteristic function of $\mathfrak{o}^m$, and $e_m = (0, \dots, 0, 1) \in F^m$. By the unramified computation of the Rankin-Selberg integrals of $\mathrm{GL}_m \times \mathrm{GL}_m$, we have that

$$[W^\circ, W^{\circ\vee}] = L(1, \tau_m \times \tau_m^\vee) = L(1, \tau_m, \mathrm{Ad}).$$

Therefore, if $\langle \cdot, \cdot \rangle$ is a non-degenerate $\mathrm{GL}_m(F)$-invariant pairing $\tau_m \times \tau_m^\vee \to \mathbb{C}$, satisfying for unramified τ_m that $\langle W^\circ, W^{\circ\vee} \rangle = 1$, then $\langle W, W^\vee \rangle = \frac{1}{L(1,\tau_m,\mathrm{Ad})} [W, W^\vee]$. We set for any irreducible generic unitarizable τ_m, $W \in \mathcal{W}(\tau_m, \psi)$, $W^\vee \in \mathcal{W}(\tau_m^\vee, \psi)$, the pairing

$$\langle W, W^\vee \rangle = \frac{1}{L(1, \tau_m, \mathrm{Ad})} [W, W^\vee].$$

Theorem 3.2 ([17, Lemma 4.7])**.** *Let $1 \le i \le m$. Let U_i be the unipotent radical of the parabolic subgroup of $\mathrm{GL}_m(F)$ of type $(m-i, 1, \dots, 1)$. Let N_i be the upper unipotent subgroup of $\mathrm{GL}_i(F)$. Then*

$$\begin{aligned} &\int_{U_{i-1}} [\pi(u_{i-1}) W, W^\vee] \, \psi(u_{i-1}) \mathrm{d}u_{i-1} \\ &= \int_{N_{m-i} \backslash \mathrm{GL}_{m-i}(F)} W \begin{pmatrix} g_{m-i} & \\ & I_i \end{pmatrix} W^\vee \begin{pmatrix} g_{m-i} & \\ & I_i \end{pmatrix} \left|\det g_{m-i}\right|^{1-i} \mathrm{d}g_{m-i}, \end{aligned}$$

for every $W \in \mathcal{W}(\tau_m, \psi)$, $W^\vee \in \mathcal{W}(\tau_m^\vee, \psi^{-1})$.

We now consider the integral

$$\alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}; W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}}) = \int_{\mathrm{GL}_n(F)} \langle \pi_{n,\underline{a}}(h_n)W_n^{\underline{a}}, W_n^{\vee\underline{a}}\rangle \langle \pi_{n+1,\underline{b}}(h_n)W_{n+1}^{\underline{b}}, W_{n+1}^{\vee\underline{b}}\rangle \mathrm{d}h_n,$$

where $W_n^{\underline{a}} \in \mathcal{W}(\pi_{n,\underline{a}}, \psi)$, $W_{n+1}^{\underline{b}} \in \mathcal{W}(\pi_{n+1,\underline{b}}, \psi)$, $W_n^{\vee\underline{a}} \in \mathcal{W}(\pi_{n,\underline{a}}^\vee, \psi^{-1})$, $W_{n+1}^{\vee\underline{b}} \in \mathcal{W}(\pi_{n+1,\underline{b}}^\vee, \psi^{-1})$. Then, by Theorem 3.2 with $m = n+1$, $i = 1$, we have

$$\begin{aligned}(3.1)\qquad & L(1, \pi_{n,\underline{a}}, \mathrm{Ad})L(1, \pi_{n+1,\underline{b}}, \mathrm{Ad})\alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}; W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}}) \\ &= \int_{\mathrm{GL}_n(F)} [\pi_{n,\underline{a}}(h_n)W_n^{\underline{a}}, W_n^{\vee\underline{a}}] \int_{N_n\backslash \mathrm{GL}_n(F)} W_{n+1}^{\underline{b}} \begin{pmatrix} g_n^\vee h_n & \\ & 1\end{pmatrix} W_{n+1}^{\vee\underline{b}} \begin{pmatrix} g_n^\vee & \\ & 1\end{pmatrix} \mathrm{d}g_n^\vee \mathrm{d}h_n.\end{aligned}$$

Changing variables, $h_n = (g_n^\vee)^{-1}g_n$, we have that eq. (3.1) equals

$$\begin{aligned}&\int_{\mathrm{GL}_n(F)} \int_{N_n\backslash \mathrm{GL}_n(F)} \left[\pi_{n,\underline{a}}(g_n)W_n^{\underline{a}}, \pi_{n,\underline{a}}^\vee(g_n^\vee)W_n^{\vee\underline{a}}\right] W_{n+1}^{\underline{b}} \begin{pmatrix} g_n & \\ & 1\end{pmatrix} W_{n+1}^{\vee\underline{b}} \begin{pmatrix} g_n^\vee & \\ & 1\end{pmatrix} \mathrm{d}g_n^\vee \mathrm{d}g_n \\ &= \int_{N_n\backslash \mathrm{GL}_n(F)} \int_{N_n\backslash \mathrm{GL}_n(F)} \int_{N_n} \left[\pi_{n,\underline{a}}(u_n g_n)W_n^{\underline{a}}, \pi_{n,\underline{a}}^\vee(g_n^\vee)W_n^{\vee\underline{a}}\right] \psi^{-1}(u_n) \\ &\times W_{n+1}^{\underline{b}} \begin{pmatrix} g_n & \\ & 1\end{pmatrix} W_{n+1}^{\vee\underline{b}} \begin{pmatrix} g_n^\vee & \\ & 1\end{pmatrix} \mathrm{d}u_n \mathrm{d}g_n \mathrm{d}g_n^\vee.\end{aligned}$$

Using Theorem 3.2 again, this time with $m = n$, $i = n$, we get

$$\int_{N_n} \left[\pi_{n,\underline{a}}(u_n g_n)W_n^{\underline{a}}, \pi_{n,\underline{a}}^\vee(g_n^\vee)W_n^{\vee\underline{a}}\right] \psi^{-1}(u_n)\mathrm{d}u_n = W_n^{\underline{a}}(g_n)W_n^{\vee\underline{a}}(g_n^\vee).$$

Hence, we have that

$$\begin{aligned}(3.2)\qquad & \alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}; W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}}) \\ &= \frac{1}{L(1, \pi_{n,\underline{a}}, \mathrm{Ad})L(1, \pi_{n+1,\underline{b}}, \mathrm{Ad})} \int_{N_n\backslash \mathrm{GL}_n(F)} W_{n+1}^{\underline{b}} \begin{pmatrix} g_n & \\ & 1\end{pmatrix} W_n^{\underline{a}}(g_n)\mathrm{d}g_n \\ &\times \int_{N_n\backslash \mathrm{GL}_n(F)} W_{n+1}^{\vee\underline{b}} \begin{pmatrix} g_n^\vee & \\ & 1\end{pmatrix} W_n^{\vee\underline{a}}(g_n^\vee)\mathrm{d}g_n^\vee.\end{aligned}$$

The integrals in eq. (3.2) are the Rankin-Selberg integrals for $\pi_{n+1,\underline{b}} \times \pi_{n,\underline{a}}$ and $\pi_{n+1,\underline{b}}^\vee \times \pi_{n,\underline{a}}^\vee$, respectively, evaluated at $s = \frac{1}{2}$. They converge for $\mathrm{Re}\,\underline{a}$, $\mathrm{Re}\,\underline{b}$ large, and understood elsewhere using meromorphic continuation. Let $W_n^{\underline{a}}, W_{n+1}^{\underline{b}}, W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}}$ be Whittaker functions that correspond to holomorphic sections. Then the Whittaker functions are also holomorphic at every point of the group, as functions of $\underline{a}$ or $\underline{b}$, see [6, Section 3.1]. For such functions, the quotient

$$\frac{\alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}; W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}})}{L(\frac{1}{2}, \mathrm{BC}(\pi_{n+1,\underline{b}}) \times \mathrm{BC}(\pi_{n,\underline{a}}))} = \frac{\alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}; W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}})}{L(\frac{1}{2}, \pi_{n+1,\underline{b}} \times \pi_{n,\underline{a}})L(\frac{1}{2}, \pi_{n+1,\underline{b}}^\vee \times \pi_{n,\underline{a}}^\vee)}$$

is holomorphic, as a function of $\underline{a}$, $\underline{b}$, whenever $\pi_{n+1,\underline{b}}, \pi_{n,\underline{a}}$ are irreducible and generic.

We remark that this is only a formal computation. In order to make it rigorous, one needs to take care of convergence issues. However, recall that by [1], the space

$$\mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi_{n+1,\underline{b}} \otimes \pi_{n,\underline{a}}, 1) \times \mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi_{n+1,\underline{b}}^\vee \otimes \pi_{n,\underline{a}}^\vee, 1)$$

is at most one dimensional, whenever $\pi_{n+1,\underline{b}}$, $\pi_{n,\underline{a}}$ are irreducible. We can define a distinguished element of this space, by setting

$$\alpha^{\natural}_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}; W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}}) = \frac{I_{\frac{1}{2}}(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}) I^{\vee}_{\frac{1}{2}}(W_n^{\vee\underline{a}}, W_{n+1}^{\vee\underline{b}})}{L(1, \pi_{n,\underline{a}}, \mathrm{Ad}) L(1, \pi_{n+1,\underline{b}}, \mathrm{Ad})},$$

where

$$I_s(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}) = \frac{1}{L(s, \pi_{n+1,\underline{b}} \times \pi_{n,\underline{a}})} \int_{N_n \backslash \mathrm{GL}_n(F)} W_{n+1}^{\underline{b}} \begin{pmatrix} g_n & \\ & 1 \end{pmatrix} W_n^{\underline{a}}(g_n) \left|\det g_n\right|^{s-\frac{1}{2}} \mathrm{d}g_n,$$

$$I^{\vee}_s(W_n^{\underline{a}}, W_{n+1}^{\underline{b}}) = \frac{1}{L(s, \pi^{\vee}_{n+1,\underline{b}} \times \pi^{\vee}_{n,\underline{a}})} \int_{N_n \backslash \mathrm{GL}_n(F)} W_{n+1}^{\vee\underline{b}} \begin{pmatrix} g_n & \\ & 1 \end{pmatrix} W_n^{\vee\underline{a}}(g_n) \left|\det g_n\right|^{s-\frac{1}{2}} \mathrm{d}g_n,$$

where the integrals converge for $\mathrm{Re}\, s$ large, and understood elsewhere by holomorphic continuation.

3.2.3. *Mixed L-factors.* Before stating our main result, we define a formal naive mixed L-factor, which will serve as a candidate of the denominator for our statement. The reason for defining this naive factor is that for some values of $\underline{a}, \underline{b}$, the representations $\pi_{n,\underline{a}}$,$\pi_{n+1,\underline{b}}$ might not be irreducible.

We denote

$$\begin{aligned} L(\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}}) = & \prod_{i=1}^{l} \prod_{j=1}^{l'} \prod_{\varepsilon,\varepsilon' \in \{\pm 1\}} L(\tfrac{1}{2}, \omega_i^{\varepsilon} \mu_j^{\varepsilon'} \left|\cdot\right|^{\varepsilon a_i + \varepsilon' b_j}) \\ & \times \prod_{i=1}^{l} L_{\mathrm{PSR}}(a_i + \tfrac{1}{2}, \pi_{m'} \times \omega_i) L_{\mathrm{PSR}}(-a_i + \tfrac{1}{2}, \pi^{\vee}_{m'} \times \omega_i^{\vee}) \\ & \times \prod_{j=1}^{l'} L_{\mathrm{PSR}}(b_j + \tfrac{1}{2}, \pi_m \times \mu_j) L_{\mathrm{PSR}}(-b_j + \tfrac{1}{2}, \pi^{\vee}_m \times \mu_j^{\vee}). \end{aligned}$$

By using [27, Proposition 7.1] repeatedly, we have that if we consider irreducible unramified representations $\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}}$ (where we take $\max(m, m') = 1$ and $\pi_{\max(m,m')}$ is the trivial representation), then

$$L(\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}}) = L(\tfrac{1}{2}, \mathrm{BC}(\pi_{n,\underline{a}}) \times \mathrm{BC}(\pi_{n+1,\underline{b}})).$$

Let $\chi \left|\cdot\right|^s$ be a character of $E^{\times}$, and let $\pi_{n+2,(s,\underline{a})} = \chi \left|\cdot\right|^s \times \pi_{n,\underline{a}}$ as in Section 2.4. Denote

$$L(\left|\cdot\right|^s \chi, \pi_{n+1,\underline{b}}) = L_{\mathrm{PSR}}(s + \tfrac{1}{2}, \pi_{m'} \times \chi) \prod_{j=1}^{l} \prod_{\varepsilon' \in \{\pm 1\}} L(s + \tfrac{1}{2}, \chi \mu_j^{\varepsilon'} \left|\cdot\right|^{\varepsilon' b_j}),$$

$$L(\left|\cdot\right|^{-s} \chi^{-1}, \pi^{\vee}_{n+1,\underline{b}}) = L_{\mathrm{PSR}}(-s + \tfrac{1}{2}, \pi^{\vee}_{m'} \times \chi^{-1}) \prod_{j=1}^{l} \prod_{\varepsilon' \in \{\pm 1\}} L(-s + \tfrac{1}{2}, \chi^{-1} \mu_j^{\varepsilon'} \left|\cdot\right|^{\varepsilon' b_j}).$$

Then we have

$$L(\pi_{n+1,\underline{b}}, \pi_{n+2,(s,\underline{a})}) = L(\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}}) L(\left|\cdot\right|^s \chi, \pi_{n+1,\underline{b}}) L(\left|\cdot\right|^{-s} \chi^{-1}, \pi^{\vee}_{n+1,\underline{b}}). \tag{3.3}$$

Consider the doubling integrals for $\mathbf{V} = V_{n+1}$, $\pi = \pi_{n+1,\underline{b}}$ discussed in Section 2.3. By [27, Section 6], for every fixed $\underline{b}$, $f_\rho^s \in I(\chi\left|\cdot\right|^s, V_{n+1})$, $f_{n+1}^{\underline{b}} \in \pi_{n+1,\underline{b}}$, $f_{n+1}^{\vee\underline{b}} \in \pi_{n+1,\underline{b}}^\vee$, we have that

$$\frac{Z(f_\rho^s, f_{n+1}^{\underline{b}}, f_{n+1}^{\vee\underline{b}})}{L(\left|\cdot\right|^s\chi, \pi_{n+1,\underline{b}})} \in \mathbb{C}[q^{\pm s}].$$

Suppose that $\left|\cdot\right|^t$ is an unramified character of $E^\times$. Similarly to the discussion above, by considering the doubling integrals for $\mathbf{V} = V_{n+1}$, $\pi = \pi_{n+1,\underline{b}}^\vee$, we have that for every fixed $\underline{b}$, $f_{\rho^\vee}^{-t} \in I(\chi^{-1}\left|\cdot\right|^{-t}, V_{n+1})$, $f_{n+1}^{\vee\underline{b}} \in \pi_{n+1,\underline{b}}^\vee$, $f_{n+1}^{\underline{b}} \in \pi_{n+1,\underline{b}}$, the following quotient is polynomial:

$$\frac{Z(f_{\rho^\vee}^{-t}, f_{n+1}^{\vee\underline{b}}, f_{n+1}^{\underline{b}})}{L(\left|\cdot\right|^{-t}\chi^{-1}, \pi_{n+1,\underline{b}}^\vee)} \in \mathbb{C}[q^{\pm t}].$$

3.2.4. *Statement of the main result .* We are now ready to state our main result. The proof of this result will be given in Section 3.4.

Theorem 3.3. *For every holomorphic sections $f_n^{\underline{a}} \in \pi_{n,\underline{a}}$, $f_{n+1}^{\underline{b}} \in \pi_{n+1,\underline{b}}$, $f_n^{\vee\underline{a}} \in \pi_{n,\underline{a}}^\vee$, $f_{n+1}^{\vee\underline{b}} \in \pi_{n+1,\underline{b}}^\vee$, the map*

$$(\underline{a}, \underline{b}) \mapsto \alpha^\natural_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}) = \frac{\alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}})}{L(\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}})},$$

originally defined only for imaginary $\underline{a}, \underline{b}$, has an analytic continuation to the entire plane. The analytic continuation is actually a polynomial, i.e., an element of $\mathbb{C}[q^{\pm\underline{a}}, q^{\pm\underline{b}}]$.

Recall that we are interested in an extension of the normalized functional $\mathcal{P}_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}$. As explained previously in Section 3.2.3, when $\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}}$ are unramified and irreducible, we have that $L(\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}}) = L(\frac{1}{2}, \mathrm{BC}(\pi_{n,\underline{a}}) \times \mathrm{BC}(\pi_{n+1,\underline{b}}))$. If $\left\|\mathrm{Re}\,\underline{a}\right\|, \left\|\mathrm{Re}\,\underline{b}\right\| < \frac{1}{2}$, and $\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}}$ are irreducible, we have that the assignments $\underline{a} \mapsto L(1, \pi_{n,\underline{a}}, \mathrm{Ad})$, $\underline{b} \mapsto L(1, \pi_{n+1,\underline{b}}, \mathrm{Ad})$ are holomorphic. In the case that $\left\|\mathrm{Re}\,\underline{a}\right\|, \left\|\mathrm{Re}\,\underline{b}\right\| < \frac{1}{2}$, and that $\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}}$ are unramified, we have that $\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}}$ are irreducible, and that

$$\begin{aligned}\mathcal{P}_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}) &= \Delta_{n+1}^{-1} \cdot L(1, \pi_{n,\underline{a}}, \mathrm{Ad}) L(1, \pi_{n+1,\underline{b}}, \mathrm{Ad}) \\ &\quad \times \alpha^\natural_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}),\end{aligned}$$

and hence we have a holomorphic extension for $\mathcal{P}_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}$.

When $\left\|\mathrm{Re}\,\underline{a}\right\|, \left\|\mathrm{Re}\,\underline{b}\right\| < \frac{1}{2}$, and when π_m, $\pi_{m'}$ are arbitrary irreducible tempered representations, we expect that whenever $\pi_{n,\underline{a}}$, $\pi_{n+1,\underline{b}}$ are irreducible we should have:

$$L(\tfrac{1}{2}, \mathrm{BC}(\pi_{n,\underline{a}}) \times \mathrm{BC}(\pi_{n+1,\underline{b}})) = L(\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}}) L(\tfrac{1}{2}, \mathrm{BC}(\pi_m) \times \mathrm{BC}(\pi_{m'})).$$

The problem is that the factor $L(\frac{1}{2}, \mathrm{BC}(\pi_m) \times \mathrm{BC}(\pi_{m'}))$ is not defined. However, since π_m, $\pi_{m'}$ are tempered representations, $L(s, \mathrm{BC}(\pi_m) \times \mathrm{BC}(\pi_{m'}))$ should be holomorphic at $\mathrm{Re}\,s > 0$, and therefore we have that $L(\frac{1}{2}, \mathrm{BC}(\pi_{n,\underline{a}}) \times \mathrm{BC}(\pi_{n+1,\underline{b}}))$ is equal to $L(\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}})$ up to a non-zero scalar. Therefore, one should still be able to define $\mathcal{P}_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}$ in this case.

Lastly, we remark that the above discussion was under the assumption that the L-factors of base change are compatible with the L-factors arising from the doubling method, i.e., for $l \in \{m, m'\}$, and a unitary character of $E^\times$, we have

$$L(s + \tfrac{1}{2}, \mathrm{BC}(\pi_l) \times \chi) = L_{\mathrm{PSR}}(s + \tfrac{1}{2}, \pi_l \times \chi). \tag{3.4}$$

However, this compatibility is not very important: since π_l is tempered, we expect the left hand side of eq. (3.4) to be holomorphic for $\mathrm{Re}\, s > -\frac{1}{2}$, and this is true for the right hand side of eq. (3.4) by [27, Lemma 7.2], hence we have that $L(\frac{1}{2}, \mathrm{BC}(\pi_{n,\underline{a}}) \times \mathrm{BC}(\pi_{n+1,\underline{b}}))$ is the same as $L(\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}})$, up to a rational function in $\mathbb{C}(q^{\pm \underline{a}}, q^{\pm \underline{b}})$, which is holomorphic and does not vanish in the region $\|\mathrm{Re}\, \underline{a}\|, \|\mathrm{Re}\, \underline{b}\| < \frac{1}{2}$.

3.3. **A lemma about $\Lambda_{f_\rho^s, v_{n+1}}$.** In this section, we relate the Rankin-Selberg integrals from Section 2.4 to the L-factors arising from the doubling method from Section 2.3. This relation will be essential for the proof of our main result.

We use the same notations as in Section 2.5. Let $\mathcal{K}_{n+2} \subset \mathrm{U}(V_{n+2})$ be a maximal compact subgroup in good position with respect to Q.

Lemma 3.4. *For $g_{n+2} \in \mathrm{U}(V_{n+2})$, the following are equivalent:*

(1) $i(g_{n+2}, \mathrm{id}_{V_n}) \in P_{V_{n+1}^\Delta}$.
(2) $g_{n+2} \in Q$ and g_{n+2} has trivial $\mathrm{U}(V_n)$ Levi part.

Proof. Let $p = i(g_{n+2}, \mathrm{id}_{V_n})$. Write $g_{n+2}(b + \overline{b}) = w_n + \lambda b + \lambda' \overline{b}$, where $w_n \in V_n$, $\lambda, \lambda' \in E$. Then

$$p(b + \overline{b}) = g_{n+2}(b + \overline{b}) = w_n + \lambda b + \lambda' \overline{b}.$$

We have that $p(b + \overline{b}) \in V_{n+1}^\Delta$ if and only if $w_n = 0$ and $\lambda = \lambda'$. This is equivalent to $g_{n+2}(b + \overline{b}) = \lambda(b + \overline{b})$, for some $\lambda \in E^\times$, which by definition is equivalent to $g_{n+2} \in Q$.

Let $v_n \in V_n$. Then

$$p(v_n + \overline{v}_n) = g_{n+2} v_n + \overline{v}_n.$$

By writing $g_{n+2} v_n = v_n' + \lambda b + \lambda' \overline{b}$, where $v_n' \in V_n$, $\lambda, \lambda' \in E$, we get that $p(v_n + \overline{v}_n) \in V_{n+1}^\Delta$, if and only if $v_n' + \lambda b = v_n + \lambda' b$, which is equivalent to $v_n = v_n'$ and $\lambda' = \lambda$. Hence, we get that $p(v_n + \overline{v}_n) \in V_{n+1}^\Delta$ if and only if $g_{n+2} v_n = v_n + \lambda(b + \overline{b})$, for some $\lambda \in E$. This is equivalent to saying that the $\mathrm{U}(V_n)$ Levi part of g_{n+2} is trivial.

Therefore, we have shown that $i(g_{n+2}, g_n)$ preserves the subspace

$$\mathbb{C}(b + \overline{b}) \oplus \mathrm{span}_{\mathbb{C}} \{v_n + \overline{v_n} \mid v_n \in V_{n+1}\} = V_{n+1}^\Delta$$

if and only if $g_{n+2} \in Q$ and g_{n+2} has trivial $\mathrm{U}(V_n)$ Levi part. □

As an immediate consequence of the lemma, we get the following corollary:

Corollary 3.5. *Let $K_0 \subset \mathrm{U}(V_{n+2})$ be a compact open subgroup. Let $g_{n+2} \in \mathrm{U}(V_{n+2})$, such that $i(g_{n+2}, \mathrm{id}_{V_n}) \in P_{V_{n+1}^\Delta} i(K_0 \times \{\mathrm{id}_{V_n}\})$. Then $g_{n+2} = q \cdot k_0$ for some $q \in Q$ having trivial $\mathrm{U}(V_n)$ Levi part, and $k_0 \in K_0$.*

The following lemma shows that holomorphic sections of $\pi_{n+2,s}$ can be represented as finite sums of elements of the form $\Lambda_{f_\rho^s, v_n}$.

Lemma 3.6. *For every standard section (with respect to $\mathcal{K}_{n+2}$) $f_{n+2}^s \in \pi_{n+2}^s$, there exist holomorphic sections $(f_{\rho,i}^s)_{i=1}^N \subset \rho_{\chi,s}$ and vectors $(v_{n,i})_{i=1}^N \subset \pi_n$, such that*

$$f_{n+2}^s = \sum_{j=1}^N \Lambda_{f_{\rho,j}^s, v_{n,j}}.$$

Proof. Denote by $\ell_{Q,E^\times} : Q \to E^\times$, $\ell_{Q,\mathrm{U}(V_n)} : Q \to \mathrm{U}(V_n)$ the projections of Q on its Levi parts. Let $K_0 \subset \mathcal{K}_{n+2}$ be a normal compact open subgroup, such that $\chi\left|\cdot\right|^s$ is trivial on $\ell_{Q,E^\times}(Q \cap K_0)$, and let $v_0 \in \pi_n$, such that v_0 is invariant to the π_n action of $\ell_{Q,\mathrm{U}(V_n)}(Q \cap K_0)$. Consider the section $f_{n+2}^{K_0,v_0,s}$ defined by ($q \in Q$, $k \in \mathcal{K}_{n+2}$):

$$f_{n+2}^{K_0,v_0,s}(q \cdot k) = \begin{cases} \delta_Q^{\frac{1}{2}}(q)(\chi\left|\cdot\right|^s \boxtimes \pi_n)(q)v_0 & k \in K_0, \\ 0 & k \notin K_0. \end{cases}$$

Every standard section of $\pi_{n+2,s}$ is a finite sum of $\mathcal{K}_{n+2}$ right translations of sections of the form $f_{n+2}^{K_0,v_0,s}$. We notice that for $h_{n+2} \in \mathrm{U}(V_{n+2})$

$$\Lambda_{\rho_{\chi,s}(i(h_{n+2},1))f_\rho^s,v_0} = \pi_{n+2}^s(h_{n+2})\Lambda_{f_\rho^s,v_0}.$$

Hence, it suffices to prove that every section of the form $f_{n+2}^{K_0,v_0,s}$ can be represented as some $\Lambda_{\rho_{\chi,s}f_\rho^s,v_0'}$, for some holomorphic section $f_\rho^s \in \rho_{\chi,s}$ and some $v_0' \in \pi_n$.

Let $f_{n+2}^{K_0,v_0,s}$ be a section as above. Since the orbit

$$P_{V_{n+1}^\Delta} \cdot i(\mathrm{U}(V_{n+1}) \times \mathrm{U}(V_{n+1})) = P_{V_{n+1}^\Delta} \cdot i(\mathrm{U}(V_{n+1}) \times \{\mathrm{id}_{V_{n+1}}\})$$

is open, it follows that $P_{V_{n+1}^\Delta} \cdot i(K_0 \times \{\mathrm{id}_{V_n}\})$ is open. Let f_ρ^s be the section of $\rho_{\chi,s}$, which is supported on $P_{V_{n+1}^\Delta} \cdot i(K_0 \times \{\mathrm{id}_{V_n}\})$, whose value on $i(K_0 \times \{\mathrm{id}_{V_n}\})$ is 1. Then for $g_{n+2} \in \mathrm{U}(V_{n+2})$ we have

$$\Lambda_{f_\rho^s,v_0}(g_{n+2}) = \int_{\mathrm{U}(V_n)} f_\rho^s(i(g_n^{-1}g_{n+2}, \mathrm{id}_{V_n}))\pi_n(g_n)v_0 \mathrm{d}g_n. \tag{3.5}$$

In order for this integral not to vanish, we must have that $i(g_n^{-1}g_{n+2}, \mathrm{id}_{V_n}) \in P_{V_{n+1}^\Delta} \cdot i(K_0 \times \{\mathrm{id}_{V_n}\})$, for some $g_n \in \mathrm{U}(V_n)$. By Corollary 3.5, this is equivalent to $g_n^{-1}g_{n+2} = q \cdot k_0$ for some $q \in Q$ having trivial $\mathrm{U}(V_n)$ Levi part, and some $k_0 \in K_0$. Since $\mathrm{U}(V_n) \subset Q$, we get that if the integral in eq. (3.5) doesn't vanish, then $g_{n+2} = q' \cdot k_0$, where $q' \in Q$ and $k_0 \in K_0$.

Now suppose $g_{n+2} = q \cdot k_0$, where $q \in Q$, and $k_0 \in K_0$. Since $\Lambda_{f_\rho^s,v_0} \in \pi_{n+2}^s$, we have

$$\Lambda_{f_\rho^s,v_0}(g_{n+2}) = \Lambda_{f_\rho^s,v_0}(q \cdot k_0) = \delta_Q^{\frac{1}{2}}(q)(\chi\left|\cdot\right|^s \boxtimes \pi_n)(q)\Lambda_{f_\rho^s,v_0}(k_0).$$

Write

$$\Lambda_{f_\rho^s,v_0}(k_0) = \int_{\mathrm{U}(V_n)} f_\rho^s(i(g_n^{-1} \cdot k_0, \mathrm{id}_{V_n}))\pi_n(g_n)v_0 \mathrm{d}g_n. \tag{3.6}$$

By its construction, f_ρ^s is invariant to right translations of $i(K_0 \times \{\mathrm{id}_{V_n}\})$. Therefore, we may assume without loss of generality that $k_0 = \mathrm{id}_{V_{n+2}}$.

The integrand in eq. (3.6) is supported on $g_n \in \mathrm{U}(V_n)$, such that $i(g_n^{-1}, \mathrm{id}_{V_n}) \in P_{V_{n+1}^\Delta} \cdot i(K_0 \times \{\mathrm{id}_{V_n}\})$. By Corollary 3.5, the last condition is equivalent to $g_n^{-1} = q' \cdot k_0'$, where $q' \in Q$ has trivial $\mathrm{U}(V_n)$ Levi part, and $k_0' \in K_0$. This implies $(k_0')^{-1} = g_n q'$, which implies that $g_n q' \in Q \cap K_0$, and hence g_n is in $\ell_{Q,\mathrm{U}(V_n)}(Q \cap K_0)$.

Conversely, suppose that $g_n \in \ell_{Q,\mathrm{U}(V_n)}(Q \cap K_0)$, then there exists $q' \in Q$ with trivial $\mathrm{U}(V_n)$ Levi part such and $k_0' \in K_0$, such that $g_n q' = k_0'$. Since $g_n q' \in Q \cap K_0$, by the choice of K_0 and v_0, we have

$$\delta_Q^{\frac{1}{2}}(g_n q') \cdot (\chi\left|\cdot\right|^s \boxtimes \pi_n)(g_n q')v_0 = v_0,$$

which implies that $\pi_n(g_n)v_0 = v_0$ and $(\chi\,|\cdot|^s)(\ell_{Q,E^\times}(q')) = 1$. This implies

$$f_\rho^s(i(g_n^{-1}, \mathrm{id}_{V_n}))\pi_n(g_n)v_0 = f_\rho^s(i(q' \cdot (k_0')^{-1}, \mathrm{id}_{V_n}))v_0 = v_0,$$

where we used the fact that f_ρ^s is right invariant to $i(K_0 \times \{\mathrm{id}_{V_n}\})$, and that $\det_\Delta(q') = \ell_{Q,E^\times}(q')$.

To summarize, we get that the integral in eq. (3.6) is supported on $g_n \in \ell_{Q,\mathrm{U}(V_n)}(Q \cap K_0)$, and that for such g_n the integrand equals v_0. Therefore, we get that $\Lambda_{f_\rho^s,v_0}(k_0) = \Lambda_{f_\rho^s,v_0}(\mathrm{id}_{V_{n+2}}) = C_{K_0}\cdot v_0$, where $C_{K_0} = \mathrm{Vol}(\ell_{Q,\mathrm{U}(V_n)}(Q \cap K_0))$ is the volume of $\ell_{Q,\mathrm{U}(V_n)}(Q \cap K_0)$ in $\mathrm{U}(V_n)$. Therefore, we showed that

$$\Lambda_{f_\rho^s,v_0} = C_{K_0} \cdot f_{n+2}^{K_0,v_0,s}.$$

Since $\ell_{Q,\mathrm{U}(V_n)}(Q \cap K_0) \subset \mathrm{U}(V_n)$ is compact and open, we get that $C_{K_0} > 0$, and therefore we have that

$$C_{K_0}^{-1} \cdot \Lambda_{f_\rho^s,v_0} = f_{n+2}^{K_0,v_0,s},$$

as required. □

As a result of the lemma, we get the following corollary:

Corollary 3.7. *For every holomorphic section $f_{n+2}^s \in \pi_{n+2,s}$, and every $v_{n+1} \in \pi_{n+1}$, there exist holomorphic sections $(f_{\rho,j}^s)_{j=1}^N \subset \rho_{\chi,s}$ and $(v_{n+1,j}^\vee)_{j=1}^N \subset \pi_{n+1}^\vee$, such that*

$$C_{\pi_{n+1},\pi_{n+2,s}}(v_{n+1}, f_{n+2}^s) = \sum_{j=1}^N Z(f_{\rho,j}^s, v_{n+1}, v_{n+1,j}^\vee),$$

where $C_{\pi_{n+1},\pi_{n+2,s}}$ is defined in Section 2.4.

Proof. By Lemma 3.6, there exist holomorphic sections $(f_{\rho,j}^s)_{j=1}^N \subset \rho_{\chi,s}$ and vectors $(v_{n,j})_{j=1}^N \subset \pi_n$, such that $f_{n+2}^s = \sum_{j=1}^N \Lambda_{f_{\rho,j}^s,v_{n,j}}$, and then

$$C_{\pi_{n+1},\pi_{n+2,s}}(v_{n+1}, f_{n+2}^s) = \sum_{j=1}^N C_{\pi_{n+1},\pi_{n+2,s}}(v_{n+1}, \Lambda_{f_{\rho,j}^s,v_{n,j}}).$$

By the identity in eq. (2.1), we have that

$$C_{\pi_{n+1},\pi_{n+2,s}}(v_{n+1}, \Lambda_{f_{\rho,j}^s,v_{n,j}}) = \int_{\mathrm{U}(V_{n+1})} f_{\rho,j}^s(i(g_{n+1}, \mathrm{id}_{V_{n+1}}))c_{\pi_n,\pi_{n+1}}(v_{n,j}, \pi_{n+1}(g_{n+1})v_{n+1})\mathrm{d}g_{n+1}. \tag{3.7}$$

By [25, Lemma 4.3], for every j, there exists $v_{n+1,j}^\vee \in \pi_{n+1}^\vee$, such that $Z(f_{\rho,j}^s, v_{n+1}, v_{n+1,j}^\vee)$ equals the right hand side of eq. (3.7), as required. □

Combining this corollary with the discussion in Section 3.2.3, we get the following result for the case $\pi_n = \pi_{n,\underline{a}}$, $\pi_{n+1} = \pi_{n+1,\underline{b}}$, $\pi_{n+2,s} = \pi_{n+2,(s,\underline{a})}$:

Proposition 3.8. *Let $\underline{a}, \underline{b}$ be fixed. Then for any holomorphic sections $f_{n+2}^{(s,\underline{a})} \in \pi_{n+2,(s,\underline{a})}$, $f_{n+1}^{\underline{b}} \in \pi_{n+1,\underline{b}}$, and any $c_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}} \in \mathrm{Hom}_{\mathrm{U}(V_n)}(\pi_{n,\underline{a}} \otimes \pi_{n+1,\underline{b}}, 1)$, we have that the following quotient is a polynomial (an element of $\mathbb{C}[q^{\pm s}]$):*

$$\frac{C_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}(f_{n+2}^{(s,\underline{a})}, f_{n+1}^{\underline{b}})}{L(|\cdot|^s\,\chi, \pi_{n+1,\underline{b}})}.$$

3.4. **Proof of the main result.** In this section, we prove Theorem 3.3. Our proof is by induction, and our construction is based on the identity in eq. (2.2). Let $f_n^{\underline{a}} \in \pi_{n,\underline{a}}$, $f_{n+1}^{\underline{b}} \in \pi_{n+1,\underline{b}}$, $f_n^{\vee\underline{a}} \in \pi_{n,\underline{a}}^\vee$, $f_{n+1}^{\vee\underline{b}} \in \pi_{n+1,\underline{b}}^\vee$ be holomorphic sections. We recall that we are looking for a holomorphic extension for the assignment

$$(\underline{a},\underline{b}) \mapsto \alpha^\natural_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}}) = \frac{\alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f_n^{\underline{a}}, f_{n+1}^{\underline{b}}; f_n^{\vee\underline{a}}, f_{n+1}^{\vee\underline{b}})}{L(\pi_{n,\underline{a}},\pi_{n+1,\underline{b}})}.$$

If $n = \min(m,m')$, $n+1 = \max(m,m')$ then there is nothing to prove, as $\pi_{n,\underline{a}} = \pi_m$, $\pi_{n+1,\underline{b}} = \pi_{m'}$, and therefore we don't have parameters and since π_m, $\pi_{m'}$ are both tempered, the assignment is defined for all vectors.

Suppose that $\alpha^\natural_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}$ is already defined. We move to construct $\alpha^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}$. Let $f_{n+1}^{\underline{b}}, f_{n+2}^{(s,\underline{a})}; f_{n+1}^{\vee\underline{b}}, f_{n+2}^{\vee(s,\underline{a})}$ be sections of $\pi_{n+1,\underline{b}}, \pi_{n+2,(s,\underline{a})}, \pi^\vee_{n+1,\underline{b}}, \pi^\vee_{n+2,(s,\underline{a})}$, respectively. If $\underline{a}$, $\underline{b}$, s are imaginary and fixed, we can use the identity in eq. (2.2), which reads

$$\begin{aligned}(3.8)\qquad &\alpha_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}(f_{n+1}^{\underline{b}}, f_{n+2}^{(s,\underline{a})}; f_{n+1}^{\vee\underline{b}}, f_{n+2}^{\vee(s,\underline{a})})\\ &= \int_{(\mathrm{U}(V_n)\backslash \mathrm{U}(V_{n+1}))^2} \mathrm{d}(g_{n+1}, g'_{n+1})\\ &\times \alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f_{n+2}^{(s,\underline{a})}(g_{n+1}), \pi_{n+1,\underline{b}}(g_{n+1})f_{n+1}^{\underline{b}}; f_{n+2}^{\vee(s,\underline{a})}(g'_{n+1}), \pi^\vee_{n+1,\underline{b}}(g'_{n+1})f_{n+1}^{\vee\underline{b}}),\end{aligned}$$

and hence by eq. (3.3)

$$\begin{aligned}(3.9)\qquad &\alpha^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}(f_{n+1}^{\underline{b}}, f_{n+2}^{(s,\underline{a})}; f_{n+1}^{\vee\underline{b}}, f_{n+2}^{\vee(s,\underline{a})})\\ &= \frac{1}{L(|\cdot|^s\chi, \pi_{n+1,\underline{b}})L(|\cdot|^{-s}\chi^{-1}, \pi^\vee_{n+1,\underline{b}})} \int_{(\mathrm{U}(V_n)\backslash \mathrm{U}(V_{n+1}))^2} \mathrm{d}(g_{n+1}, g'_{n+1})\\ &\times \alpha^\natural_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f_{n+2}^{(s,\underline{a})}(g_{n+1}), \pi_{n+1,\underline{b}}(g_{n+1})f_{n+1}^{\underline{b}}; f_{n+2}^{\vee(s,\underline{a})}(g'_{n+1}), \pi^\vee_{n+1,\underline{b}}(g'_{n+1})f_{n+1}^{\vee\underline{b}}).\end{aligned}$$

We are going to use eq. (3.9) in order to extend the definition of $\alpha^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}$.

We introduce a new variable t (representing an unramified character $|\cdot|^t$ of $E^\times$), and consider the integral

$$\begin{aligned}(3.10)\qquad &\beta^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})},\pi^\vee_{n+2,(t,\underline{a})}}(f_{n+1}^{\underline{b}}, f_{n+2}^{(s,\underline{a})}; f_{n+1}^{\vee\underline{b}}, f_{n+2}^{\vee(t,\underline{a})})\\ &= \frac{1}{L(|\cdot|^s\chi, \pi_{n+1,\underline{b}})L(|\cdot|^{-t}\chi^{-1}, \pi^\vee_{n+1,\underline{b}})} \int_{(\mathrm{U}(V_n)\backslash \mathrm{U}(V_{n+1}))^2} \mathrm{d}(g_{n+1}, g'_{n+1})\\ &\times \alpha^\natural_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f_{n+2}^{(s,\underline{a})}(g_{n+1}), \pi_{n+1,\underline{b}}(g_{n+1})f_{n+1}^{\underline{b}}; f_{n+2}^{\vee(t,\underline{a})}(g'_{n+1}), \pi^\vee_{n+1,\underline{b}}(g'_{n+1})f_{n+1}^{\vee\underline{b}}),\end{aligned}$$

where $f_{n+1}^{\underline{b}} \in \pi_{n+1,\underline{b}}, f_{n+1}^{\vee\underline{b}} \in \pi^\vee_{n+1,\underline{b}}$, $f_{n+2}^{(s,\underline{a})} \in \pi_{n+2,(s,\underline{a})}$, $f_{n+2}^{\vee(t,\underline{a})} \in \pi^\vee_{n+2,(t,\underline{a})}$ are holomorphic sections.

Proposition 3.9. *For fixed $\underline{a},\underline{b}$, the integral in eq. (3.10) absolutely converges for* $\mathrm{Re}\, s$ *large and* $\mathrm{Re}(-t)$ *large (both depending on $\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}}$). $\beta^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})},\pi^\vee_{n+2,(t,\underline{a})}}(f_{n+1}^{\underline{b}}, f_{n+2}^{(s,\underline{a})}; f_{n+1}^{\vee\underline{b}}, f_{n+2}^{\vee(t,\underline{a})})$ has a meromorphic continuation to the entire plane, which we continue to denote by the same symbol. This meromorphic continuation is actually polynomial, i.e.,*

$$\beta^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})},\pi^\vee_{n+2,(t,\underline{a})}}(f_{n+1}^{\underline{b}}, f_{n+2}^{(s,\underline{a})}; f_{n+1}^{\vee\underline{b}}, f_{n+2}^{\vee(t,\underline{a})}) \in \mathbb{C}[q^{\pm s}, q^{\pm t}].$$

Proof. If $\alpha^{\natural}_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}} = 0$, there is nothing to prove. Otherwise, we have that $\alpha^{\natural}_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}} \in \mathrm{Hom}_{\mathrm{U}(V_n)}(\pi_{n,\underline{a}} \otimes \pi_{n+1,\underline{b}}, 1) \boxtimes \mathrm{Hom}_{\mathrm{U}(V_n)}(\pi^{\vee}_{n,\underline{a}} \otimes \pi^{\vee}_{n+1,\underline{b}}, 1)$. By [1, 2], we have that the spaces $\mathrm{Hom}_{\mathrm{U}(V_n)}(\pi_{n,\underline{a}} \otimes \pi_{n+1,\underline{b}}, 1)$, $\mathrm{Hom}_{\mathrm{U}(V_n)}(\pi^{\vee}_{n,\underline{a}} \otimes \pi^{\vee}_{n+1,\underline{b}}, 1)$ are at most one-dimensional. Therefore, we can write

$$\alpha^{\natural}_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}; f^{\vee\underline{b}}_{n+1}, f^{\vee(t,\underline{a})}_{n+2}) = c_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}) \cdot c_{\pi^{\vee}_{n,\underline{a}},\pi^{\vee}_{n+1,\underline{b}}}(f^{\vee\underline{b}}_{n+1}, f^{\vee(t,\underline{a})}_{n+2}),$$

where $c_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}} \in \mathrm{Hom}_{\mathrm{U}(V_n)}(\pi_{n,\underline{a}} \otimes \pi_{n+1,\underline{b}}, 1)$, $c_{\pi^{\vee}_{n,\underline{a}},\pi^{\vee}_{n+1,\underline{b}}} \in \mathrm{Hom}_{\mathrm{U}(V_n)}(\pi^{\vee}_{n,\underline{a}} \otimes \pi^{\vee}_{n+1,\underline{b}}, 1)$. We have that the integral in eq. (3.10) is now given by

$$\begin{aligned}
&\beta^{\natural}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})},\pi^{\vee}_{n+2,(t,\underline{a})}}(f^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}; f^{\vee\underline{b}}_{n+1}, f^{\vee(t,\underline{a})}_{n+2}) \\
&= \frac{1}{L(|\cdot|^{s}\chi, \pi_{n+1,\underline{b}})L(|\cdot|^{-t}\chi^{-1}, \pi^{\vee}_{n+1,\underline{b}})} \\
&\quad \times \int_{\mathrm{U}(V_n)\backslash\mathrm{U}(V_{n+1})} c_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f^{(s,\underline{a})}_{n+2}(g_{n+1}), \pi_{n+1,\underline{b}}(g_{n+1})f^{\underline{b}}_{n+1})\mathrm{d}g_{n+1} \\
&\quad \times \int_{\mathrm{U}(V_n)\backslash\mathrm{U}(V_{n+1})} c_{\pi^{\vee}_{n,\underline{a}},\pi^{\vee}_{n+1,\underline{b}}}(f^{\vee(t,\underline{a})}_{n+2}(g'_{n+1}), \pi^{\vee}_{n+1,\underline{b}}(g'_{n+1})f^{\vee\underline{b}}_{n+1})\mathrm{d}g'_{n+1}.
\end{aligned}$$

This is a product of two Rankin-Selberg integrals, as in Section 2.4. We have that these integrals converge absolutely for $\mathrm{Re}\, s$ large and $\mathrm{Re}(-t)$ large, and have meromorphic continuations to the entire plane, which are rational functions in q^{-s}, q^{-t}, respectively. By Proposition 3.8, we have that the multiplicands

$$\frac{1}{L(|\cdot|^{s}\chi, \pi_{n+1,\underline{b}})} \int_{\mathrm{U}(V_n)\backslash\mathrm{U}(V_{n+1})} c_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(f^{(s,\underline{a})}_{n+2}(g_{n+1}), \pi_{n+1,\underline{b}}(g_{n+1})f^{\underline{b}}_{n+1})\mathrm{d}g_{n+1},$$

$$\frac{1}{L(|\cdot|^{-t}\chi^{-1}, \pi^{\vee}_{n+1,\underline{b}})} \int_{\mathrm{U}(V_n)\backslash\mathrm{U}(V_{n+1})} c_{\pi^{\vee}_{n,\underline{a}},\pi^{\vee}_{n+1,\underline{b}}}(f^{\vee(t,\underline{a})}_{n+2}(g'_{n+1}), \pi^{\vee}_{n+1,\underline{b}}(g'_{n+1})f^{\vee\underline{b}}_{n+1})\mathrm{d}g'_{n+1}$$

are elements of $\mathbb{C}[q^{\pm s}]$, $\mathbb{C}[q^{\pm t}]$, respectively. Hence, we get that

$$\beta^{\natural}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})},\pi^{\vee}_{n+2,(t,\underline{a})}}(f^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}; f^{\vee\underline{b}}_{n+1}, f^{\vee(t,\underline{a})}_{n+2}) \in \mathbb{C}[q^{\pm s}, q^{\pm t}].$$

□

For fixed $\underline{a}, \underline{b}$, we have shown that $\beta^{\#}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})},\pi^{\vee}_{n+2,(t,\underline{a})}}(f^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}; f^{\vee\underline{b}}_{n+1}, f^{\vee(t,\underline{a})}_{n+2})$ is a polynomial in $q^{\pm s}, q^{\pm t}$. We may now substitute $s = t$ and denote

$$\beta^{\natural}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}(f^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}; f^{\vee\underline{b}}_{n+1}, f^{\vee(s,\underline{a})}_{n+2}) = \beta^{\natural}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})},\pi^{\vee}_{n+2,(s,\underline{a})}}(f^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}; f^{\vee\underline{b}}_{n+1}, f^{\vee(s,\underline{a})}_{n+2}).$$

Our next task to is to show that $\beta^{\natural}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}$ is an extension of $\alpha^{\natural}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}$. In order to show this, we need the following lemma:

Lemma 3.10. *Suppose that there exist holomorphic sections* $\varphi^{\underline{a}}_{n} \in \pi_{n,\underline{a}}$, $\varphi^{\underline{b}}_{n+1} \in \pi_{n+1,\underline{b}}$, $\varphi^{\vee\underline{a}}_{n} \in \pi^{\vee}_{n,\underline{a}}$, $\varphi^{\vee\underline{b}}_{n+1} \in \pi^{\vee}_{n+1,\underline{b}}$, *such that* $\alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(\varphi^{\underline{a}}_{n}, \varphi^{\underline{b}}_{n+1}; \varphi^{\vee\underline{a}}_{n}, \varphi^{\vee\underline{b}}_{n+1}) = 1$, *for every* $\underline{a}, \underline{b}$ *imaginary. Then there exist holomorphic sections* $\varphi^{(s,\underline{a})}_{n+2} \in \pi_{n+2,(s,\underline{a})}$, $\varphi^{\vee(s,\underline{a})}_{n+2} \in \pi^{\vee}_{n+2,(s,\underline{a})}$, *such that* $\alpha_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}(\varphi^{\underline{b}}_{n+1}, \varphi^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, \varphi^{\vee(s,\underline{a})}_{n+2}) = 1$, *for all* $\underline{a}$, $\underline{b}$, s *imaginary, and* $\beta^{\natural}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}(\varphi^{\underline{b}}_{n+1}, \varphi^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, \varphi^{\vee(s,\underline{a})}_{n+2}) = L(\pi_{n+1,\underline{b}}, \pi_{n+2,(s,\underline{a})})^{-1}$, *for and all* $\underline{a}, \underline{b}$, *imaginary and every* s.

Proof. We realize $\pi_{n+2,(s,\underline{a})}$ and $\pi^\vee_{n+2,(s,\underline{a})}$ as in Section 3.1.1.

When all parameters are imaginary, all representations are tempered, and we can use the identity in eq. (3.8). The integral in eq. (3.8) absolutely converges in this case [3, Claim (7.4.10)].

Let $K_0 \subset \mathrm{U}(V_{n+1})$ be a compact open subgroup, such that $\varphi^{\underline{b}}_{n+1}$, $\varphi^{\vee\underline{b}}_{n+1}$ are invariant under its action, and so that $\varphi^{\underline{a}}_n$ (respectively $\varphi^{\vee\underline{a}}_n$) is invariant under the $\chi\left|\cdot\right|^s \boxtimes \pi_n$ action (respectively $\chi^{-1}\left|\cdot\right|^{-s}\boxtimes\pi^\vee_n$ action) of $K_0 \cap P_{Ef_+,\mathrm{U}(V_{n+2})}$. We have that $\mathrm{U}(V_{n+1})P_{Ef_+,\mathrm{U}(V_{n+2})}$ is a dense open subset of $\mathrm{U}(V_{n+2})$, and hence so is $P_{Ef_+,\mathrm{U}(V_{n+2})}\mathrm{U}(V_{n+1})$. This implies that $P_{Ef_+,\mathrm{U}(V_{n+2})}K_0$ is an open subset of $\mathrm{U}(V_{n+2})$. Let $f^{(s,\underline{a})}_{n+2}, f^{\vee(s,\underline{a})}_{n+2}$ be sections of $\pi_{n+2,(s,\underline{a})}, \pi^\vee_{n+2,(s,\underline{a})}$, supported on the open subset $P_{Ef_+,\mathrm{U}(V_{n+2})}K_0$, such that their restriction to K_0 is $\varphi^{\underline{a}}_n, \varphi^{\vee\underline{a}}_n$ respectively. Then we have that $f^{(s,\underline{a})}_{n+2}, f^{\vee(s,\underline{a})}_{n+2}$ are holomorphic sections, and by eq. (3.8), we get that

$$\alpha_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}(\varphi^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, f^{\vee(s,\underline{a})}_{n+2}) = C^2_{K_0}\cdot\alpha_{\pi_{n,\underline{a}},\pi_{n+1,\underline{b}}}(\varphi^{\underline{a}}_n, \varphi^{\underline{b}}_{n+1}; \varphi^{\vee\underline{a}}_n, \varphi^{\vee\underline{b}}_{n+1}) = C^2_{K_0},$$

where $C_{K_0} = \mathrm{Vol}(\mathrm{U}(V_n)\backslash\mathrm{U}(V_n)K_0)$ is the volume of $\mathrm{U}(V_n)\backslash\mathrm{U}(V_n)K_0$ inside $\mathrm{U}(V_n)\backslash\mathrm{U}(V_{n+1})$. Therefore, by choosing $\varphi^{(s,\underline{a})}_{n+2} = \frac{1}{C_{K_0}}f^{(s,\underline{a})}_{n+2}$, $\varphi^{\vee(s,\underline{a})}_{n+2} = \frac{1}{C_{K_0}}f^{\vee(s,\underline{a})}_{n+2}$, we get the desired result.

Similarly, we have that $\beta^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})},\pi^\vee_{n+2,(t,\underline{a})}}(\varphi^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, f^{\vee(t,\underline{a})}_{n+2})$ is given by eq. (3.10), for $\mathrm{Re}\, s$, $\mathrm{Re}(-t)$ large. We get by the construction of $f^{(s,\underline{a})}_{n+2}$, $f^{\vee(t,\underline{a})}_{n+2}$ that

$$\begin{aligned}&\beta^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})},\pi^\vee_{n+2,(t,\underline{a})}}(\varphi^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, f^{\vee(t,\underline{a})}_{n+2})\\ &= \frac{1}{L(\left|\cdot\right|^s\chi, \pi_{n+1,\underline{b}})L(\left|\cdot\right|^{-t}\chi^{-1}, \pi^\vee_{n+1,\underline{b}})}\frac{1}{L(\pi_{n,\underline{a}}, \pi_{n+1,\underline{b}})}C^2_{K_0},\end{aligned}$$

which implies that for all s, $\beta^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}(\varphi^{\underline{b}}_{n+1}, \varphi^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, \varphi^{\vee(s,\underline{a})}_{n+2}) = L(\pi_{n+1,\underline{b}}, \pi_{n+2,(s,\underline{a})})^{-1}$. ☐

Using the last lemma repeatedly, we are able to conclude the following:

Corollary 3.11. *Let $m_0 = \min(m, m')$, $m_0 + 1 = \max(m, m')$. Then $\alpha^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}$ is not identically zero if and only if $\alpha_{\pi_{m_0},\pi_{m_0+1}} \neq 0$, and we have that $\alpha^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}} = \beta^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}$, if s, $\underline{a}$, $\underline{b}$ are imaginary. Furthermore, for holomorphic sections $f^{\underline{b}}_{n+1} \in \pi_{n+1,\underline{b}}$, $f^{(s,\underline{a})}_{n+2} \in \pi_{n+2,(s,\underline{a})}$, $f^{\vee\underline{b}}_{n+1} \in \pi^\vee_{n+1,\underline{b}}$, $f^{\vee(s,\underline{a})}_{n+2} \in \pi^\vee_{n+2,(s,\underline{a})}$, we have that $\beta^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}(f^{\underline{b}}_{n+1}, f^{(s,\underline{a})}_{n+2}, f^{\vee\underline{b}}_{n+1}, f^{\vee(s,\underline{a})}_{n+2}) \in \mathbb{C}[q^{\pm\underline{a}}, q^{\pm\underline{b}}, q^{\pm s}]$.*

Proof. If $\alpha_{\pi_{m_0},\pi_{m_0+1}} = 0$, then we get by repeatedly using the recursive formula in eq. (3.8) that $\alpha^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}} = 0$, and that $\beta^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}} = 0$. Suppose that $\alpha_{\pi_{m_0},\pi_{m_0+1}} \neq 0$. Then we can find vectors $\varphi_{m_0} \in \pi_{m_0}$, $\varphi_{m_0+1} \in \pi_{m_0+1}$, $\varphi^\vee_{m_0} \in \pi^\vee_{m_0}$, $\varphi^\vee_{m_0+1} \in \pi^\vee_{m_0+1}$, such that $\alpha_{\pi_{m_0},\pi_{m_0+1}}(\varphi_{m_0}, \varphi_{m_0+1}; \varphi^\vee_{m_0}, \varphi^\vee_{m_0+1}) = 1$. Repeatedly using Lemma 3.10, we get that we can find sections $\varphi^{\underline{b}}_{n+1} \in \pi_{n+1,\underline{b}}$, $\varphi^{\vee\underline{b}}_{n+1} \in \pi^\vee_{n+1,\underline{b}}$, $\varphi^{(s,\underline{a})}_{n+2} \in \pi_{n+2,(s,\underline{a})}$, $\varphi^{\vee(s,\underline{a})}_{n+2} \in \pi^\vee_{n+2,(s,\underline{a})}$, such that for every $\underline{a}, \underline{b}$ imaginary,

$$\begin{aligned}\alpha^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}(\varphi^{\underline{b}}_{n+1}, \varphi^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, \varphi^{\vee(s,\underline{a})}_{n+2}) &= \beta^\natural_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}(\varphi^{\underline{b}}_{n+1}, \varphi^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, \varphi^{\vee(s,\underline{a})}_{n+2})\\ &= L(\pi_{n+1,\underline{b}}, \pi_{n+2,(s,\underline{a})})^{-1}.\end{aligned}$$

We have that both $\alpha^{\natural}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}$, $\beta^{\natural}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}$ define elements of

$$\mathrm{Hom}_{\mathrm{U}(V_n)}(\pi_{n+1,\underline{b}} \otimes \pi_{n+2,(s,\underline{a})}, 1) \boxtimes \mathrm{Hom}_{\mathrm{U}(V_n)}(\pi^{\vee}_{n+1,\underline{b}} \otimes \pi^{\vee}_{n+2,(s,\underline{a})}, 1). \tag{3.11}$$

The representations $\pi_{n+1,\underline{b}}$, $\pi_{n+2,(s,\underline{a})}$ are irreducible for $\underline{a}, \underline{b}, s$ outside of a finite union of hyperplanes, and then the space in eq. (3.11) is at most one dimensional [2]. Therefore, we must have that when $\underline{a}, \underline{b}, s$ are imaginary, $\alpha^{\natural}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}} = \beta^{\natural}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}$.

By Bernstein's rationality theorem (see for example [6, Section 3.2]), the above discussion shows that

$$\beta^{\natural}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}(\varphi^{\underline{b}}_{n+1}, \varphi^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, \varphi^{\vee(s,\underline{a})}_{n+2}) \in \mathbb{C}(q^{\pm\underline{a}}, q^{\pm\underline{b}}, q^{\pm s}).$$

Since for every fixed $\underline{a}$, $\underline{b}$, we have that

$$\beta^{\natural}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}(\varphi^{\underline{b}}_{n+1}, \varphi^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, \varphi^{\vee(s,\underline{a})}_{n+2}) \in \mathbb{C}[q^{\pm s}],$$

we must have that

$$\beta^{\natural}_{\pi_{n+1,\underline{b}},\pi_{n+2,(s,\underline{a})}}(\varphi^{\underline{b}}_{n+1}, \varphi^{(s,\underline{a})}_{n+2}; \varphi^{\vee\underline{b}}_{n+1}, \varphi^{\vee(s,\underline{a})}_{n+2}) \in \mathbb{C}[q^{\pm\underline{a}}, q^{\pm\underline{b}}, q^{\pm s}].$$

□

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