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On exterior square gamma functions for representations of GL_{2m}

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Introduction

Let F be a non-archimedean local field. Let π be a smooth irreducible representation of $\mathrm{GL}_n(F)$. By the local Langlands correspondence there exists an nth dimensional representation $\rho(\pi)$ of the Weil-Deligne group W'_F associated to π. The local exterior square L-function of π is defined via this correspondence as $L(s, \pi, \wedge^2) = L(s, \wedge^2(\rho(\pi)))$. We will be only interested in the case where n is even.

In [JS90], Jacquet and Shalika study the global exterior square L-function for irreducible automorphic cuspidal representations on GL_n, mainly for the case where n is even. In Section 7 of [JS90], Jacquet and Shalika give an integral representation for the local exterior square L-function, for unramified irreducible representations of $\mathrm{GL}_{2m}(F)$. On the other hand, in [Sha90] in Section 7, Shahidi proposes another potential construction for this L-function, via the Langlands-Shahidi method. In [KR12], Kewat and Raghunathan show that these three constructions for the local exterior square L-function agree, for all smooth irreducible representations of $\mathrm{GL}_{2m}(F)$ [KR12, Theorem 1.4].

In [Mat14], Matringe proves the corresponding local functional equation. This functional equation is already proved by Kewat and Raghunathan in their paper [KR12] using global arguments. Matringe's proof uses only local arguments.

In this work, we discuss the local non-archimedean theory corresponding to the Jacquet-Shalika integral mentioned above. In Theorems A-D mentioned below, we give a survey for known results of this theory. We follow the proofs of Jacquet and Shalika, and of Matringe, and add details to the original proofs. Our contributions are the theories and the theorems that appear after Theorem D, although these might be known to the experts of the field.

We now present the main theorems that we prove.

The theory over a p-adic field. Let F be a p-adic field. Let π be an irreducible smooth generic representation of $\mathrm{GL}_{2m}(F)$.

Theorem (A). *There exists $r_{\pi,\wedge^2} \in \mathbb{R}$, such that for every $s \in \mathbb{C}$, with $\mathrm{Re}(s) > r_{\pi,\wedge^2}$, $W \in \mathcal{W}(\pi,\psi)$, $\phi \in \mathcal{S}(F^m)$, the following integral converges absolutely*

$$J_{\pi,\psi}(s, W, \phi) = \int_{N\backslash \mathrm{GL}_m(F)} \int_{\mathcal{B}\backslash M_m(F)} W\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right)\psi(-\mathrm{tr}X)\, dX \cdot \phi(\varepsilon g)\left|\det g\right|^s dg.$$

Theorem (B). *There exist $W \in \mathcal{W}(\pi,\psi)$, $\phi \in \mathcal{S}(F^m)$, such that for every $s \in \mathbb{C}$, with $\mathrm{Re}(s) > r_{\pi,\wedge^2}$,*

$$J_{\pi,\psi}(s, W, \phi) = 1.$$

We follow the proofs of Jacquet and Shalika in [JS90, Sections 7.1, 7.3] for Theorems A and B.

Theorem (C). *For a fixed $W \in \mathcal{W}(\pi,\psi)$, $\phi \in \mathcal{S}(F^m)$, the function $J_{\pi,\psi}(s, W, \phi)$ results in an element of $\mathbb{C}(q^{-s})$ in the convergence domain, and therefore has a meromorphic continuation. Furthermore, denote*

$$I_{\pi,\psi} = \mathrm{span}_{\mathbb{C}}\left\{J_{\pi,\psi}(s, W, \phi) \mid W \in \mathcal{W}(\pi,\psi), \phi \in \mathcal{S}(F^m)\right\},$$

then there exists a unique $p(z) \in \mathbb{C}[z]$, such that $p(0) = 1$ and $I_{\pi,\psi} = \frac{1}{p(q^{-s})}\mathbb{C}[q^{-s}, q^s]$. $p(z)$ does not depend on ψ. We denote $L(s, \pi, \wedge^2) = \frac{1}{p(q^{-s})}$.

Kewat and Raghunathan denote $L_{JS}(s,\pi,\wedge^2)=\frac{1}{p(q^{-s})}$, and show that every smooth irreducible generic representation π, $L_{JS}(s,\pi,\wedge^2)$ is the same function as the one constructed via the local Langlands correspondence (this is shown by Jacquet and Shalika only for unramified representations).

As a result of Theorem C, $J_{\pi,\psi}(s,W,\phi)$ has a meromorphic continuation to the entire complex plane, which we keep to denote as $J_{\pi,\psi}(s,W,\phi)$.

Assume from now and on that π is supercuspidal. We prove the following theorems.

Theorem (D)**.** *There exists an element $\gamma_{\pi,\psi}(s)\in\mathbb{C}(q^{-s})$, such that for every $\phi\in\mathcal{S}(F^m)$, $W\in\mathcal{W}(\pi,\psi)$, one has*

$$J_{\tilde{\pi},\psi^{-1}}\left(1-s,\tilde{\pi}\left(\begin{pmatrix} & I_m\\ I_m & \end{pmatrix}\right)\tilde{W},\hat{\phi}\right)=\gamma_{\pi,\psi}(s)\cdot J_{\pi,\psi}(s,W,\phi).$$

Furthermore,

$$\gamma_{\pi,\psi}(s)=\varepsilon_{\pi,\psi}(s)\cdot\frac{L(1-s,\tilde{\pi},\wedge^2)}{L(s,\pi,\wedge^2)},$$

where $\varepsilon_{\pi,\psi}(s)$ is an invertible element of $\mathbb{C}[q^{-s},q^s]$.

We follow the proof of Matringe [Mat12, Mat14] for Theorem D.

Theorem (E)**.** *The following are equivalent.*

(1) *$\omega_\pi\equiv 1$ and there exists $W\in\mathcal{W}(\pi,\psi)$, such that*

$$l_{\pi,\psi}(W)=\int_{ZN\backslash \mathrm{GL}_m(F)}\int_{\mathcal{B}\backslash M_m(F)}W\left(w_{m,m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}g & \\ & g\end{pmatrix}\right)\psi(-\mathrm{tr}X)\,dX\,dg\neq 0.$$

(2) *$\gamma_{\pi,\psi}(s)$ has a pole at $s=1$.*

(3) *$L(s,\pi,\wedge^2)$ has a pole at $s=0$.*

We prove Theorem E using the functional equation, which was discussed in Theorem D. A variation of this theorem is already known for Shahidi's construction of the exterior square L function (see the introduction of [JNQ08] and Theorem 5.5 of the same paper).

Theorem (F)**.** *If ω_π is ramified, then $L(s,\pi,\wedge^2)=L(ms,\omega_\pi)=1$. If ω_π is unramified then*

$$L\left(s,\pi,\wedge^2\right)=\prod_{k\in S_{\pi,\psi}}\frac{1}{1-\omega_\pi(\varpi)^{\frac{1}{m}}\zeta^k q^{-s}},$$

where $\zeta=e^{\frac{2\pi i}{m}}$ and

$$\begin{aligned}S_{\pi,\psi}=&\Big\{0\le k\le m-1\mid \exists W\in\mathcal{W}(\pi,\psi),\\ &\int_{ZN\backslash G}\left(\int_{\mathcal{B}\backslash M}W\left(w_{m,m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}g & \\ & g\end{pmatrix}\right)\psi(-\mathrm{tr}(X))\,dX\right)\left|\det g\right|^{\frac{2\pi i k-\log\omega_\pi(\varpi)}{m\log q}}dg\neq 0\Big\}.\end{aligned}$$

The theory over a finite field. We also develop an analogous theory corresponding to Jacquet-Shalika integral, over a finite field $\mathbb{F}_q$. Our main results are the following.

Let π be an irreducible generic representation of $\mathrm{GL}_{2m}(\mathbb{F}_q)$.

Theorem (B'). *There exist $W \in \mathcal{W}(\pi,\psi)$ and $\phi \in \mathcal{S}\left(\mathbb{F}_q^m\right)$, such that*

$$1 = J_{\pi,\psi}(W,\phi) = \frac{1}{[\mathrm{GL}_m(\mathbb{F}_q) : N]\,[M_m(\mathbb{F}_q) : \mathcal{B}]} \sum_{g \in N \backslash \mathrm{GL}_m(\mathbb{F}_q)} \sum_{X \in \mathcal{B} \backslash M_m(\mathbb{F}_q)} W\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right) \cdot \psi(-\mathrm{tr}X) \cdot \phi(\varepsilon g).$$

Assume from now and on that π is cuspidal.

Theorem (D'). *Suppose that π does not admit a Shalika vector. Then there exists a constant $\gamma_{\pi,\psi} \in \mathbb{C}^*$, such that for every $W \in \mathcal{W}(\pi,\psi)$, $\phi \in \mathcal{S}\left(\mathbb{F}_q^m\right)$, one has*

$$\gamma_{\pi,\psi} \cdot J_{\pi,\psi}(W,\phi) = J_{\tilde{\pi},\psi^{-1}}\left(\tilde{\pi}\left(\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}\right)\tilde{W}, \hat{\phi}\right).$$

Let $\theta : \mathbb{F}_{q^{2m}}^* \to \mathbb{C}^*$ be a regular character associated with π.

Theorem (E'). *The following are equivalent:*

(1) *There exists $W \in \mathcal{W}(\pi,\psi)$, such that $J_{\pi,\psi}(W,1) \neq 0$.*
(2) *π admits a Shalika vector.*
(3) *$\theta \restriction_{\mathbb{F}_{q^m}^*} \equiv 1$.*

We give an expression for $\gamma_{\pi,\psi}$ for $m = 1, 2$, in terms of θ.

Theorem (G). *Suppose that $\theta \restriction_{\mathbb{F}_{q^m}^*} \not\equiv 1$ (i.e. π doesn't admit a Shalika vector). Then*

(1) *For $m = 1$,*

$$\gamma_{\pi,\psi}^{-1} = \sum_{a \in \mathbb{F}_q^*} \omega_\pi(a) . \psi^{\mathcal{F}}(-a).$$

(2) *For $m = 2$,*

$$\gamma_{\pi,\psi}^{-1} = T_0 - \frac{1}{q^2}\left(\sum_{a \in \mathbb{F}_q^*} \omega_\pi(a)\,\psi^{\mathcal{F}}(-a)\right)\left(\sum_{b \in \mathbb{F}_q^*}\left(\sum_{\substack{\xi \in \mathbb{F}_{q^4}^* \\ \mathrm{N}_{\mathbb{F}_{q^4}/\mathbb{F}_q}(\xi) = b^2}} \sum_{\beta \in \mathbb{F}_q^*} \psi^{-1}\left(\beta + \frac{1}{\beta}\mathrm{Tr}_{\mathbb{F}_{q^4}/\mathbb{F}_q}\left(\xi + \frac{b}{\xi}\right)\right)\theta(\xi)\right)\right),$$

where $T_0 = \begin{cases} q - \frac{1}{q} & \omega_\pi \equiv 1 \\ 0 & \omega_\pi \not\equiv 1 \end{cases}.$

Relating the theories. We conclude this work, by relating the above theories corresponding to Jacquet-Shalika integral, using level zero (depth zero) representations. Our main results are the following theorems:

Theorem (H). *Let (π_0, V_{π_0}) be an irreducible cuspidal representation of $\mathrm{GL}_{2m}(\mathbb{F}_q)$, and let π be a level zero representation of $\mathrm{GL}_{2m}(F)$, constructed through π_0. Then for every $v \in V_{\pi_0}$, $\phi \in \mathcal{S}\left(\mathbb{F}_q^m\right)$, $s \in \mathbb{C}$*

$$J_{\pi,\psi}(s, W_v, F_\phi) = J_{\pi_0,\psi_0}\left(W_v^0, \phi\right) + J_{\pi_0,\psi_0}\left(W_v^0, 1\right) \cdot \phi(0)\,\omega_\pi(\varpi) \cdot q^{-ms} L(ms, \omega_\pi).$$

As a result, we get a modified version of the functional equation for all cuspidal irreducible representation π of $\mathrm{GL}_{2m}(\mathbb{F}_q)$, regardless whether they admit a Shalika vector or not:

Theorem (D"). *There exists an element* $\gamma_{\pi,\psi}(s) \in \mathbb{C}(q^{-s})$*, such that for every* $\phi \in \mathcal{S}\left(\mathbb{F}_q^m\right)$*,* $W \in \mathcal{W}(\pi,\psi)$*,* $s \in \mathbb{C}$*, one has*

$$J_{\tilde{\pi},\psi^{-1}}\left(\tilde{\pi}\left(\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}\right)\tilde{W},\hat{\phi}\right) + J_{\pi,\psi}(W,1)\cdot\hat{\phi}(0)\cdot q^{-m(1-s)}L(m(1-s),1) = \gamma_{\pi,\psi}(s)\cdot\left(J_{\pi,\psi}(W,\phi) + J_{\pi,\psi}(W,1)\cdot\phi(0)\cdot q^{-ms}L(ms,1)\right).$$

Furthermore, if π *admits a Shalika vector then*

$$\gamma_{\pi,\psi}(s) = \frac{q^{ms}}{q^m}\frac{L(m(1-s),1)}{L(ms,1)}.$$

Otherwise, $\gamma_{\pi,\psi}(s) \in \mathbb{C}^*$.

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1. The Jacquet-Shalika integral

Towards this section, F is a finite field or a p-adic field. In the case that F is a finite field, we denote for $a \in F$, $|a| = \begin{cases} 1 & a \neq 0 \\ 0 & a = 0 \end{cases}$ the trivial absolute value. In the case that F is a p-adic field, we denote by $|a|$ the absolute value of a.

For an l-group G and a vector space V over $\mathbb{C}$, we denote by $\mathcal{S}(G, V)$ the space of Schwartz functions on G having values in V (smooth functions $f : G \to V$ with compact support). We also denote $\mathcal{S}(G) = \mathcal{S}(G, \mathbb{C})$. Note that if G is a finite group then $\mathcal{S}(G) = \{f : G \to \mathbb{C}\}$, $\mathcal{S}(G, V) = \{f : G \to V\}$.

1.1. **Preliminaries.**

1.1.1. *Whittaker model.* Let n be a positive integer, $G = \mathrm{GL}_n(F)$.

Given a non-trivial character $\psi : F \to \mathbb{C}^*$, we define a character, also denoted ψ, on the upper triangular unipotent matrix subgroup N of G by

$$\psi\left(\begin{pmatrix} 1 & a_1 & * & * & * \\ & 1 & a_2 & * & * \\ & & \ddots & \ddots & * \\ & & & 1 & a_{n-1} \\ & & & & 1 \end{pmatrix}\right) = \psi\left(\sum_{k=1}^{n-1} a_k\right).$$

Let π be a (smooth) representation of G. π is called generic if $\mathrm{Hom}_G\left(\pi, \mathrm{Ind}_N^G(\psi)\right) \neq 0$. It is known that supercuspidal (cuspidal if F is finite) representations are generic ([BZ76, Proposition 5.15.a]).

It is known that if π is irreducible and generic, then $\dim \mathrm{Hom}_G\left(\pi, \mathrm{Ind}_N^G(\psi)\right) = 1$ ([BZ76, Theorem 5.16], [Bum, Theorem 6.1]). In this case, we denote by $\mathcal{W}(\pi, \psi)$ the unique subspace of $\mathrm{Ind}_N^G(\psi)$ which is equivalent to π. This is called the Whittaker model of π with respect to ψ.

It is known that for an irreducible representation π of G, the contragredient representation $\tilde{\pi}$ is isomorphic to π^l where $\pi^l(g) = \pi\left(g^l\right)$ and $g^l = \left(g^{-1}\right)^t = \left(g^t\right)^{-1}$ ([BZ76, Theorem 7.3]).

Suppose that π is generic and irreducible. For $W \in \mathcal{W}(\pi, \psi)$ we define $\tilde{W} : G \to \mathbb{C}$ by $\tilde{W}(g) = W\left(w_n \cdot g^l\right)$ where $w_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$.

Proposition 1.1. *The image of the map $W \mapsto \tilde{W}$ is $\mathcal{W}(\tilde{\pi}, \psi^{-1})$ (the Whittaker model of $\tilde{\pi}$ in respect to the character ψ^{-1}). (where G acts on $\widetilde{\mathcal{W}(\pi, \psi)}$ by right translations. We denote this action by $\tilde{\rho}$).*

Proof. We denote the action of G on $\mathrm{Ind}_N^G(\psi)$ by ρ. Note that $\left(\tilde{\rho}(h)\tilde{W}\right)(g) = \tilde{W}(gh) = W\left(w_n \cdot g^l h^l\right) = \widetilde{\rho\left(h^l\right) W}(g) = \widetilde{\rho^l(h) W}(g)$. Therefore $W \mapsto \tilde{W}$ is a homomorphism $\tilde{\pi} \cong \pi^l \cong \rho^l \to \tilde{\rho}$. It is non-trivial and therefore its image is isomorphic to $\tilde{\pi}$. We now check that for $W \in \mathcal{W}(\pi,\psi)$, we have $\tilde{W} \in \mathrm{Ind}_N^G(\psi^{-1})$. A direct computation shows that for $u \in N$

$$u = \begin{pmatrix} 1 & a_1 & * & * & * \\ & 1 & a_2 & * & * \\ & & \ddots & \ddots & * \\ & & & 1 & a_{n-1} \\ & & & & 1 \end{pmatrix}, \quad w_n \cdot u^l \cdot w_n = \begin{pmatrix} 1 & -a_{n-1} & * & * & * \\ & 1 & -a_{n-2} & * & * \\ & & 1 & \ddots & * \\ & & & \ddots & -a_1 \\ & & & & 1 \end{pmatrix} \in N.$$

Therefore $\psi\left(w_n u^l w_n\right) = \psi^{-1}(u)$, and the proposition follows. ☐

We denote for $g,h \in G$ and $W \in \mathcal{W}(\pi,\psi)$, $(\lambda(h)W)(g) = W(h^{-1}g)$. Denote for $a \in F^*$, $\psi_a(x) = \psi(ax)$. For $W \in \mathcal{W}(\pi,\psi)$ and $a \in F^*$ denote $W^a = \lambda(\mathrm{diag}(1,a,\ldots,a^{2m-1}))W$.

Proposition 1.2. *The image of the map $W \mapsto W^a$ is $\mathcal{W}(\pi,\psi_a)$.*

Proof. It is clear that this map is a non-trivial homomorphism with respect to the action of right translations. One easily checks that its image is contained in $\mathrm{Ind}_N^G(\psi_a)$. ☐

1.1.2. *Haar measure.* Let G be an l-group. It is common knowledge that there exists a unique (up to multiplication by a positive scalar) right Haar measure which is right invariant to the action of G, i.e. there exists a measure $\mu_{r,G}$ such that

$$\int_G f(ga)\, d\mu_{r,G}(g) = \int_G f(g)\, d\mu_{r,G}(g),$$

for every Schwartz function f.

Similarly, there exists a unique left Haar measure.

Theorem 1.3. *Let K be a closed subgroup of G, both assumed unimodular. There exists a unique measure $\mu_{K\backslash G}$ invariant to right translations such that for every $f \in \mathcal{S}(G)$ we have*

$$\int_{K\backslash G} \left(\int_K f(kg)\, d\mu_K(k)\right) \mu_{K\backslash G}(g) = \int_G f(g)\, d\mu_G(g)$$

(See [Lan12, Page 37, Theorem 1]*).*

Remark 1.4. Note that the map $g \mapsto \int_K f(kg)\, d\mu_K(k)$ is constant on cosets ${}_K\backslash^G$

In the following we choose for a finite group G the following normalized Haar measure

$$\int_G f(g)\, d\mu_G(g) = \frac{1}{|G|} \sum_{g\in G} f(g),$$

and therefore we have the following Haar measure on the quotient space: for $K \leq G$ and $f : {}_K\backslash^G \to \mathbb{C}$,

$$\int_{K\backslash G} f(g)\, dg = \frac{1}{[G:K]} \sum_{g \in K\backslash G} f(g).$$

1.1.3. *Fourier transform.* Let $\psi : F \to \mathbb{C}^*$ be a non-trivial additive character of F.

It is standard knowledge that all (continuous) characters of F are of the form $\psi_a(x) = \psi(ax)$ where $a \in F$. Such a is unique.

It follows that all (continuous) characters of F^n are of the form $\psi_{\underline{a}}(\underline{x}) = \psi(\langle \underline{a}, \underline{x} \rangle)$ where $\underline{a} \in F^n$ (where $\langle \underline{a}, \underline{x} \rangle = \underline{a} \cdot \underline{x}^t = \sum_{i=1}^n a_i x_i$) and such $\underline{a}$ is unique. In the special case of $M_n(F) \cong F^{n^2}$ all (additive continuous) characters have the form $\psi_A(X) = \psi(\mathrm{tr}(A \cdot X))$ where $A \in M_n(F)$, and such A is unique.

Fix a non-trivial additive character $\psi^{\mathcal{F}} : F \to \mathbb{C}^*$.

For $G = F, F^n, M_n(F)$, the Fourier transform of a Schwartz function $f : G \to \mathbb{C}$ is defined as

$$\hat{f}(a) = \int_G f(x) \psi_a^{\mathcal{F}}(x) \, d\mu_G(x),$$

where μ_G is a Haar measure of G. It is known that μ_G can be normalized such that $\hat{\hat{f}}(a) = f(-a)$ (Fourier inversion theorem).

In the case where F is a finite field and the Haar measure is the normalized Haar measure as chosen before on G, the Fourier inversion theorem has the form

$$\hat{\hat{f}}(a) = \frac{1}{|G|} f(-a).$$

Let $f \in \mathcal{S}(F^n)$ and let $g \in \mathrm{GL}_n(F)$. Define $(\rho(g) f)(\underline{x}) = f(\underline{x} g)$.

A simple change of variables in the integral yields the following:

Proposition 1.5. $\widehat{\rho(g) f} = \frac{1}{|\det g|} \rho\left(g^l\right) \hat{f}$.

1.2. **The Jacquet-Shalika integral.** Let m be a positive integer. Let π be an irreducible generic representation of $\mathrm{GL}_{2m}(F)$, and let $\psi : F \to \mathbb{C}^*$ be a non-trivial character of the additive group F. Let $G = \mathrm{GL}_m(F)$ and let be $N \leq G$ the upper triangular unipotent subgroup. Let $M = M_m(F)$ and $\mathcal{B} \leq M$ be the upper triangular subspace. Let $\varepsilon = \varepsilon_m = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in F^{1 \times m}$. Let σ be the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & m & m+1 & m+2 & m+3 & \dots & 2m \\ 1 & 3 & 5 & \dots & 2m-1 & 2 & 4 & 6 & \dots & 2m \end{pmatrix}$$

and let $w_{m,m}$ be the column permutation matrix corresponding to σ, i.e. $w_{m,m} = P_{\sigma,\mathrm{col}} = \begin{pmatrix} e_{\sigma(1)} & e_{\sigma(2)} & \dots & e_{\sigma(n)} \end{pmatrix}$.

Remark 1.6. Note that for an arbitrary matrix $(a_{ij})_{i,j} \in M_n(F)$, and for an arbitrary permutation $\tau \in S_n$, we have $P_{\tau,\mathrm{col}}(a_{i,j}) P_{\tau,\mathrm{col}}^{-1} = \left(a_{\tau^{-1}(i)\tau^{-1}(j)}\right)_{i,j}$ and therefore $w_{m,m}(a_{i,j}) w_{m,m}^{-1} = \left(a_{\sigma^{-1}(i)\sigma^{-1}(j)}\right)_{i,j}$.

Definition 1.7 (The Jacquet-Shalika integral)**.** Let $s \in \mathbb{C}$, $W \in \mathcal{W}(\pi, \psi)$, $\phi \in \mathcal{S}(F^m)$, we define

$$J_{\pi,\psi}(s, W, \phi) = \int_{N \backslash G} \int_{\mathcal{B} \backslash M} W\left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix}\right) \psi(-\mathrm{tr} X) \, dX \cdot \phi(\varepsilon g) |\det g|^s \, dg.$$

In case that F is finite, $|\det g| = 1$ for every $g \in G$ and we omit s from the notation:

$$J_{\pi,\psi}(W,\phi) = \frac{1}{[G:N][M:\mathcal{B}]} \sum_{g \in {}_N\backslash^G} \sum_{X \in {}_{\mathcal{B}}\backslash^M} W\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right)\psi(-\mathrm{tr}X)\cdot\phi(\varepsilon g).$$

Proposition 1.8. *The integrands involved are well defined (as formal expressions).*

Proof. First we show that for a fixed $g \in G$, the function

$$f(X) = W\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right)\psi(-\mathrm{tr}(X))$$

is constant on cosets of ${}_{\mathcal{B}}\backslash^M$: If $X' = X + U$ where U is an upper triangular matrix.

$$f(X+U) = W\left(w_{m,m}\begin{pmatrix} I_m & X+U \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right)\psi(-\mathrm{tr}(X+U)).$$

Denote $U = \begin{pmatrix} a_1 & * & * \\ & \ddots & * \\ & & a_m \end{pmatrix}$, $a_1, \ldots, a_m \in F$, then

$$\psi(-\mathrm{tr}(X+U)) = \psi\left(-\sum_{k=1}^{n} a_k\right)\psi(-\mathrm{tr}(X)).$$

We calculate $w_{m,m}\left(\begin{smallmatrix} I_m & U \\ & I_m \end{smallmatrix}\right)w_{m,m}^{-1}$. For a matrix $(a_{ij})_{1\le i,j\le n}$ we have

$$w_{m,m}(a_{ij})w_{m,m}^{-1} = \left(a_{\sigma^{-1}(i),\sigma^{-1}(j)}\right)_{1\le i,j\le n}.$$

It is clear that after conjugation the diagonal is preserved. We notice that the only non-diagonal entries of $\left(\begin{smallmatrix} I_m & U \\ & I_m \end{smallmatrix}\right)$ that can be non zero after conjugation are those with $(\sigma^{-1}(i), \sigma^{-1}(j)) = (i', j'+m)$ where $1 \le i' \le j' \le m$, i.e

$$(i,j) = (\sigma(i'), \sigma(j'+m)) = (2i'-1, 2j').$$

Note that $i = 2i'-1 < 2i' \le 2j' = j$ and therefore $w_{m,m}\left(\begin{smallmatrix} I_m & U \\ & I_m \end{smallmatrix}\right)w_{m,m}^{-1}$ is an upper triangular unipotent matrix, i.e. $w_{m,m}\left(\begin{smallmatrix} I_m & U \\ & I_m \end{smallmatrix}\right)w_{m,m}^{-1} \in N_{2m}$.

Finally we compute the non-zero elements above the diagonal: these are the elements with index (i,j) with $i+1 = j$. But the above computation shows $i = 2i'-1$, $j = 2j'$ and therefore $i' = j'$ and we get that the elements above the diagonal are exactly $a_1, 0, a_2, \ldots, 0, a_m$, i.e.

$$w_{m,m}\begin{pmatrix} I_m & U \\ & I_m \end{pmatrix}w_{m,m}^{-1} = \begin{pmatrix} 1 & a_1 & * & * & * & * \\ & 1 & 0 & * & * & * \\ & & 1 & a_2 & * & * \\ & & & \ddots & 0 & * \\ & & & & 1 & a_m \\ & & & & & 1 \end{pmatrix}.$$

Therefore we have $\psi\left(w_{m,m}\left(\begin{smallmatrix} I_m & U \\ & I_m \end{smallmatrix}\right)w_{m,m}^{-1}\right) = \psi\left(\sum_{k=1}^{m} a_k\right)$. It now follows that $f(X+U) = f(X)$, as required.

We now show that the expression

$$h(g) = \int_{{}_{\mathcal{B}}\backslash^M} W\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right)\psi(-\mathrm{tr}(X))\,dX\cdot\phi(\varepsilon g)\,|\det g|^s$$

is constant on cosets of ${}_{N}\backslash^{G}$.

Let $u \in N$. We have $|\det u| = 1$, $\varepsilon u = \varepsilon$.

$$h(ug) = \int_{\mathcal{B}\backslash M} W\left(w_{m,m}\begin{pmatrix} u & \\ & u\end{pmatrix}\begin{pmatrix} I_m & u^{-1}Xu \\ & I_m\end{pmatrix}\begin{pmatrix} g & \\ & g\end{pmatrix}\right)\psi\left(-\mathrm{tr}\left(X\right)\right)dX \cdot \phi\left(\varepsilon g\right)\left|\det g\right|^{s}.$$

The automorphism $X \mapsto u^{-1}Xu$ preserves the upper triangular matrix group. We substitute $X' = u^{-1}Xu$, $dX' = dX$ and $\mathrm{tr}X = \mathrm{tr}X'$.

Finally we compute $w_{m,m}\left(\begin{smallmatrix} u & \\ & u\end{smallmatrix}\right)w_{m,m}^{-1}$. As before, the diagonal is preserved under conjugation and the only non-diagonal elements of the conjugation which can be non zero are those having index $\left(\sigma^{-1}(i), \sigma^{-1}(j)\right) = (i', j')$ with $1 \le i' < j' \le m$ or $\left(\sigma^{-1}(i), \sigma^{-1}(j)\right) = (i'+m, j'+m)$ i.e. $(i,j) = (2i'-1, 2j'-1)$ or $(i,j) = (2i', 2j')$. Since $i' < j'$ we have $2i'-1 < 2j'-1$ and $2i' < 2j'$ and therefore in both cases $i < j$. Therefore $w_{m,m}\left(\begin{smallmatrix} u & \\ & u\end{smallmatrix}\right)w_{m,m}^{-1}$ is an upper triangular unipotent matrix, i.e. $w_{m,m}\left(\begin{smallmatrix} u & \\ & u\end{smallmatrix}\right)w_{m,m}^{-1} \in N$.

We check again the elements above the diagonal: these are elements having index (i,j) with $i+1 = j$. Since in the first case, both i and j are odd, and in the second case both i and j are even, we conclude that all elements above the diagonal are zero, and therefore $\psi\left(w_{m,m}\left(\begin{smallmatrix} u & \\ & u\end{smallmatrix}\right)w_{m,m}^{-1}\right) = 1$, and we conclude that $h(ug) = h(g)$, as required. □

For a finite field there is no question regarding the integral's convergence. We show in Subsection 3.2 that for a p-adic field F, the integral converges for $\mathrm{Re}(s)$ sufficiently large (larger than $r_{\pi,\wedge^2} \in \mathbb{R}$ where $r_{\pi,\wedge^2}$ depends on π only).

1.2.1. *The dual Jacquet-Shalika integral.* Let π be a generic irreducible representation of $\mathrm{GL}_{2m}(F)$, and let $s \in \mathbb{C}$, $W \in \mathcal{W}(\pi,\psi)$, $\phi \in \mathcal{S}(F^m)$, we define

$$\tilde{J}_{\pi,\psi}(s, W, \phi) = J_{\tilde{\pi},\psi^{-1}}\left(1-s, \tilde{\pi}\left(\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}\right)\tilde{W}, \hat{\phi}\right).$$

(See Subsections 1.1.1, 1.1.3).

We develop an expression for $\tilde{J}_{\pi,\psi}(s, W, \phi)$ which will be useful later.

Recalling that $\tilde{W}(g) = W\left(w_{2m}g^{l}\right)$ we get

$$\tilde{J}_{\pi,\psi}(s, W, \phi) = \int_{N\backslash G}\int_{\mathcal{B}\backslash M}\tilde{W}\left(w_{2m}w_{m,m}^{l}\begin{pmatrix} I_m & X \\ & I_m\end{pmatrix}^{l}\begin{pmatrix} g & \\ & g\end{pmatrix}^{l}\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}^{l}\right)\psi\left(\mathrm{tr}X\right)dX \cdot$$
$$\cdot\, \hat{\phi}\left(\varepsilon g\right)\left|\det g\right|^{1-s}dg,$$

a direct computation shows that

$$w_{2m}w_{m,m}^{l}\begin{pmatrix} I & X \\ & I\end{pmatrix}^{l}\begin{pmatrix} g & \\ & g\end{pmatrix}^{l}\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}^{l} = w_{2m}w_{m,m}\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}\begin{pmatrix} I_m & -X^{t} \\ & I_m\end{pmatrix}\begin{pmatrix} g^{l} & \\ & g^{l}\end{pmatrix}.$$

To proceed, we claim that w_{2m} and $w_{m,m}$ commute: it suffices to show that the permutations $\tau = \left(\begin{smallmatrix} 1 & 2 & \dots & 2m \\ 2m & 2m-1 & \dots & 1\end{smallmatrix}\right)$ and σ commute, as $w_{2m} = P_{\tau,\mathrm{col}}$, $w_{m,m} = P_{\sigma,\mathrm{col}}$ and $P_{\tau,\mathrm{col}}P_{\sigma,\mathrm{col}} = P_{\tau\circ\sigma,\mathrm{col}}$.

If $1 \le i \le m$ then $\tau(\sigma(i)) = \tau(2i-1) = 2m-(2i-1)+1 = 2m-2i+2$ and $\sigma(\tau(i)) = \sigma(2m-i+1)$. Here $2m-i+1 = m+(m+1-i) > m$ as $i < m+1$, and therefore $\sigma(2m-i+1) = 2(m+1-i) = 2m-2i+2$.

If $1 \le i \le m$ then $\tau(\sigma(i+m)) = \tau(2i) = 2m-2i+1$ and $\sigma(\tau(i+m)) = \sigma(2m-(i+m)+1) = \sigma(m-i+1)$. Here $1 \le m-i+1 \le m$ as $1 \le i \le m$, and therefore $\sigma(m-i+1) = 2(m-i+1)-1 = 2m-2i+1$.

Using the fact that $w_{m,m}$ and w_{2m} commute, and that $w_{2m} = \left(\begin{smallmatrix} & w_m \\ w_m & \end{smallmatrix}\right)$, we get by a direct computation that

$$\tilde{J}_{\pi,\psi}(s,W,\phi) = \int_{N\backslash G}\int_{\mathcal{B}\backslash M} W\left(w_{m,m}\begin{pmatrix} I_m & -w_m X^t w_m \\ & I_m \end{pmatrix}\begin{pmatrix} w_m g^l & \\ & w_m g^l \end{pmatrix}\right)\psi(\mathrm{tr}X)\,dX\cdot$$
$$\cdot\,\hat{\phi}(\varepsilon g)\left|\det g\right|^{1-s} dg.$$

Substituting $X = -w_m Y^t w_m$ and $g = w_m h^l$, we get $\mathrm{tr}X = -\mathrm{tr}Y$, $\left|\det g\right| = \left|\det h\right|^{-1}$ and $\varepsilon w_m h^l = \varepsilon_1 h^l$, where $\varepsilon_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$. Therefore

$$\tilde{J}_{\pi,\psi}(s,W,\phi) = \int_{N\backslash G}\int_{\mathcal{B}\backslash M} W\left(w_{m,m}\begin{pmatrix} I_m & Y \\ & I_m \end{pmatrix}\begin{pmatrix} h & \\ & h \end{pmatrix}\right)\psi(-\mathrm{tr}Y)\,dY\cdot\hat{\phi}\left(\varepsilon_1 h^l\right)\left|\det h\right|^{s-1} dh.$$

1.2.2. *Equivariance properties.*

Definition 1.9 (The Shalika subgroup)**.**

$$S_{2m} = \left\{\begin{pmatrix} g & X \\ & g \end{pmatrix} \mid g \in \mathrm{GL}_m(F), X \in M_m(F)\right\}$$

We define a character Ψ on the Shalika subgroup by $\Psi\left(\left(\begin{smallmatrix} g & X \\ & g \end{smallmatrix}\right)\right) = \psi\left(\mathrm{tr}\left(g^{-1}X\right)\right)$. One easily checks that this is indeed a character.

We define an action of S_{2m} on $\mathcal{S}(F^m)$ by $\left(\rho\left(\left(\begin{smallmatrix} g & X \\ & g \end{smallmatrix}\right)\right)\phi\right)(x) = \phi(xg) = (\rho(g))(x)$.

Let $s \in \mathbb{C}$, such that $J_{\pi,\psi}(s,W,\phi)$ converges (respectively such that $\tilde{J}_{\pi,\psi}(s,W,\phi)$ converges) for every $W \in \mathcal{W}(\pi,\psi)$, $\phi \in \mathcal{S}(F^m)$.

Proposition 1.10. *The map $B_s : \mathcal{W}(\pi,\psi)\times\mathcal{S}(F^m) \to \mathbb{C}$, $B_s(W,\phi) = J_{\pi,\psi}(s,W,\phi)$ (respectively $B_s(W,\phi) = \tilde{J}_{\pi,\psi}(s,W,\phi)$) is a bilinear form which is $\left|\det\right|^{-\frac{s}{2}}\cdot\Psi$ equivariant over S_{2m}, i.e. for every $\left(\begin{smallmatrix} g & X \\ & g \end{smallmatrix}\right) \in S_{2m}$, $W \in \mathcal{W}(\pi,\psi)$ and $\phi \in \mathcal{S}(F^m)$ one has*

$$B_s\left(\pi\left(\begin{pmatrix} g & X \\ & g \end{pmatrix}\right)W, \rho(g)\phi\right) = \left|\det g\right|^{-s}\psi\left(\mathrm{tr}\left(g^{-1}X\right)\right)\cdot B_s(W,\phi).$$

Proof. It suffices to prove the claim for elements of the form $\left(\begin{smallmatrix} I_m & Y \\ & I_m \end{smallmatrix}\right)$ and of the form $\left(\begin{smallmatrix} h & \\ & h \end{smallmatrix}\right)$.

For elements of the form $\left(\begin{smallmatrix} I_m & Y \\ & I_m \end{smallmatrix}\right)$ we have

$$J_{\pi,\psi}\left(s,\pi\left(\begin{pmatrix} I_m & Y \\ & I_m \end{pmatrix}\right)W,\phi\right) = \int_{N\backslash G}\int_{\mathcal{B}\backslash M} W\left(w_{m,m}\begin{pmatrix} I_m & gYg^{-1}+X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right)\psi(-\mathrm{tr}(X))\,dX\cdot$$
$$\cdot\,\phi(\varepsilon g)\left|\det g\right|^{s} dg.$$

Substituting $X' = X + gYg^{-1}$, $dX' = dX$ and $\mathrm{tr}(X) = \mathrm{tr}(X') - \mathrm{tr}(Y)$, yields the requested result.

For elements of the form $\left(\begin{smallmatrix} h & \\ & h \end{smallmatrix}\right)$ we get the result immediately by substituting $gh = g'$, $\left|\det g\right|^s = \left|\det g'\right|^s\left|\det h\right|^{-s}$.

We now show the statement for the bilinear form $B_s(W,\phi) = \tilde{J}_{\pi,\psi}(s,W,\phi)$. We use the expression

$$\tilde{J}_{\pi,\psi}(s,W,\phi) = \int_{N\backslash G}\int_{\mathcal{B}\backslash M} W\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right)\psi(-\mathrm{tr}X)\,dX\cdot\hat{\phi}\left(\varepsilon_1 g^l\right)\left|\det g\right|^{s-1} dg.$$

For elements of the form $\begin{pmatrix} I_m & Y \\ & I_m \end{pmatrix}$ the proof is exactly as before.

We check the equivariance of $\tilde{J}_{\pi,\psi}$ for elements of the form $\begin{pmatrix} h & \\ & h \end{pmatrix}$: we recall that from Proposition 1.5 we have $\widehat{\rho(h)\phi} = \frac{1}{|\det h|}\rho(h^l)\hat{\phi}$, and therefore $\tilde{J}_{\pi,\psi}\left(s, \pi\left(\begin{pmatrix} h & \\ & h \end{pmatrix}\right) W, \rho(h)\phi\right)$ equals

$$\frac{1}{|\det h|}\int_{N\backslash G}\int_{\mathcal{B}\backslash M} W\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\begin{pmatrix} h & \\ & h \end{pmatrix}\right)\psi(-\mathrm{tr}X)\,dX\cdot$$
$$\cdot\,\hat{\phi}\left(\varepsilon_1 g^l h^l\right)|\det g|^{s-1}\,dg.$$

As before, substituting $gh = g'$ yields

$$\tilde{J}_{\pi,\psi}\left(s, \pi\left(\begin{pmatrix} h & \\ & h \end{pmatrix}\right) W, \rho(h)\phi\right) = \frac{|\det h|^{1-s}}{|\det h|}\tilde{J}_{\pi,\psi}(s, W, \phi),$$

as required. □

1.2.3. *Change of the character ψ.* As noted in Subsection 1.1.3, given a non-trivial character $\psi : F \to \mathbb{C}^*$, any other non-trivial character of $\psi' : F \to \mathbb{C}^*$ is given by $\psi'(x) = \psi_a(x) = \psi(ax)$, where $a \in F^*$.

Let $a \in F^*$. We wish to relate between $J_{\pi,\psi}(s, W, \phi)$ and $J_{\pi,\psi_a}(s, W^a, \phi)$ (See also Proposition 1.2).

$$J_{\pi,\psi_a}(s, W^a, \phi) = \int_{N\backslash G}\int_{\mathcal{B}\backslash M} W\left(\mathrm{diag}\left(1, a, \dots, a^{2m-1}\right)^{-1} w_{m,m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right)\cdot$$
$$\cdot\,\psi(-a\mathrm{tr}X)\,dX\cdot\phi(\varepsilon g)\,|\det g|^s\,dg.$$

After conjugating with $w_{m,m}$ we get $w_{m,m}^{-1}\mathrm{diag}\left(1, a, \dots, a^{2m-1}\right)^{-1} w_{m,m} = \begin{pmatrix} d_a^{-1} & \\ & a^{-1}d_a^{-1} \end{pmatrix}$, where $d_a = \mathrm{diag}\left(1, a^2, \dots, a^{2m-2}\right)$. After further conjugations we get

$$\int_{N\backslash G}\int_{\mathcal{B}\backslash M} W\left(w_{m,m}\begin{pmatrix} I_m & d_a^{-1}Xd_a a \\ & I_m \end{pmatrix}\begin{pmatrix} d_a^{-1}g & \\ & d_a^{-1}g \end{pmatrix}\begin{pmatrix} I_m & \\ & a^{-1}I_m \end{pmatrix}\right)\psi(-a\mathrm{tr}X)\,dX\cdot\phi(\varepsilon g)\,|\det g|^s\,dg.$$

Replacing $d_a^{-1}g = g'$, $d_a X d_a^{-1} a = X'$, $|\det g| = |\det g'|\cdot|a|^{2\binom{m}{2}}$, $dX' = |a|^{-2\binom{m+1}{3}+\binom{m}{2}}\,dX$ (as $\sum_{1\le j<i\le m}(i-j) = \binom{m+1}{3}$), we get

$$J_{\pi,\psi_a}(s, W^a, \phi) = |a|^{2\binom{m+1}{3}+\binom{m}{2}(2s-1)}\,J_{\pi,\psi}\left(s, \pi\left(\begin{pmatrix} I_m & \\ & a^{-1}I_m \end{pmatrix}\right) W, \phi_{a^{2m-2}}\right),$$

where $\phi_{a^{2m-2}}(x) = \phi\left(a^{2m-2}\cdot x\right)$ for $x \in F^m$.

Replacing $g'' = a^{2m-2}g'$, we get the relation

$$J_{\pi,\psi_a}(s, W^a, \phi) = |a|^{\frac{m(m-1)(2m-1)}{6}}\,\omega_\pi(a)^{-2(m-1)}\,|a|^{-m(m-1)s}\,J_{\pi,\psi}\left(s, \pi\left(\begin{pmatrix} I_m & \\ & a^{-1}I_m \end{pmatrix}\right) W, \phi\right).$$

Similarly, repeating these steps for the expression of $\tilde{J}_{\pi,\psi_a}(s, W^a, \phi)$ (except of the substitution $g'' = a^{2m-2}g'$, which is not needed) yields

$$\tilde{J}_{\pi,\psi_a}(s, W^a, \phi) = |a|^{\frac{m(m-1)(2m-1)}{6}}\,|a|^{m(m-1)(s-1)}\,\tilde{J}_{\pi,\psi}\left(s, \pi\left(\begin{pmatrix} I_m & \\ & a^{-1}I_m \end{pmatrix}\right) W, \phi\right).$$

2. The Jacquet-Shalika integral over a finite field

In this section, F is a finite field and $\psi : F \to \mathbb{C}$ is a fixed non-trivial character of the additive group F.

2.1. **Preliminaries.**

2.1.1. *The Bessel function.* Let n be a positive integer and let (π, V_π) be a generic irreducible representation of $G = \mathrm{GL}_n(F)$. Therefore there exists a non-zero functional $T : V_\pi \to \mathbb{C}$ such that $\langle T, \pi(u) v\rangle = \psi(u) \langle T, v\rangle$ for every $u \in N = N_n(F)$, and $v \in V_\pi$. This functional is unique up to multiplication by a constant.

Since G is finite and π is irreducible, V_π is finite dimensional and therefore there exists an inner product $(\cdot,\cdot)$ on V_π, with respect to which π is unitary. There also exists a unique $0 \neq v_0 \in V_\pi$ such that $(v, v_0) = \langle T, v\rangle$ for every $v \in V_\pi$ which implies $\pi(u) v_0 = \psi(u) v_0$ for every $u \in N$.

The Bessel function of π with respect to ψ is defined as $\mathcal{B}_{\pi,\psi}(g) = \frac{(\pi(g)v_0,v_0)}{(v_0,v_0)}$. $\mathcal{B}_{\pi,\psi}(g)$ does not depend on the choice of T as $\dim \mathrm{Hom}_N(\pi \restriction_N, \psi) = 1$.

The Bessel function is a Whittaker function $\mathcal{B}_{\pi,\psi} \in \mathcal{W}(\pi, \psi)$, and satisfies $\mathcal{B}_{\pi,\psi}(I_n) = 1$. It also satisfies for every $g \in G$ and $u_1, u_2 \in N$, $\mathcal{B}_{\pi,\psi}(u_1 g u_2) = \psi(u_1 u_2) \mathcal{B}_{\pi,\psi}(g)$.

Proposition 2.1. [Gel70, Proposition 4.5] *The Bessel function is also given by the formula*

$$\mathcal{B}_{\pi,\psi}(g) = \frac{1}{|N|} \sum_{u \in N} \mathrm{tr}(\pi(gu)) \psi^{-1}(u).$$

Proposition 2.2. [Gel70, Proposition 4.9] *Suppose that $\mathcal{B}_{\pi,\psi}(wd) \neq 0$, where w is a permutation matrix, and d is a diagonal matrix. Then*

$$wd = \begin{pmatrix} & & & \lambda_1 I_{n_1} \\ & & \lambda_2 I_{n_2} & \\ & \iddots & & \\ \lambda_r I_{n_r} & & & \end{pmatrix},$$

where $n_1 + \cdots + n_r = n$ and $\lambda_1, \ldots, \lambda_r \in F^*$.

Corollary 2.3. *Let $g \in G$. By the Bruhat decomposition we can write $g = u_1 w d u_2$ where $u_1, u_2 \in N$, w is a permutation matrix, and d is a diagonal matrix. $\mathcal{B}_{\pi,\psi}(g) = \mathcal{B}_{\pi,\psi}(u_1 w d u_2) = \psi(u_1 u_2) \mathcal{B}_{\pi,\psi}(wd)$. Therefore if $\mathcal{B}_{\pi,\psi}(g) \neq 0$, then*

$$g = u_1 \begin{pmatrix} & & & \lambda_1 I_{n_1} \\ & & \lambda_2 I_{n_2} & \\ & \iddots & & \\ \lambda_r I_{n_r} & & & \end{pmatrix} u_2,$$

where $u_1, u_2 \in N$, and $n_1 + \cdots + n_r = n$ and $\lambda_1, \ldots, \lambda_r \in F^*$.

2.2. **Non-vanishing.** Let $n=2m$ be a positive even integer. Let π be a generic representation of $\mathrm{GL}_{2m}(F)$.

We prove that the bilinear form $J_{\pi,\psi}:\mathcal{W}(\pi,\psi)\times\mathcal{S}(F^m)\to\mathbb{C}$ is non-trivial. We use the Bessel function in the proof. One can avoid this by repeating the proof for the non-vanishment of the Jacquet-Shalika integral for the case of a p-adic field, which we give in Subsection 3.3. The following calculation will be useful in the sequel.

Proposition 2.4. *Let $\phi=\delta_\varepsilon : F^m\to\mathbb{C}$ be the indicator function of $\varepsilon=\begin{pmatrix}0 & \dots & 0 & 1\end{pmatrix}\in F^{1\times m}$, i.e. $\delta_\varepsilon(v)=\begin{cases}1 & v=\varepsilon\\ 0 & v\neq\varepsilon\end{cases}$ and let $W(g)=[G:N][M:\mathcal{B}]\mathcal{B}_{\pi,\psi}\left(g\cdot w_{m,m}^{-1}\right)$. Then $J_{\pi,\psi}(W,\phi)=1$.*

Proof. We write

$$J_{\pi,\psi}(W,\phi)=\sum_{\substack{g\in{}_N\backslash^G\\ \varepsilon g=\varepsilon}}\sum_{X\in{}_{\mathcal{B}}\backslash^M}\mathcal{B}_{\pi,\psi}\left(w_{m,m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}g & \\ & g\end{pmatrix}w_{m,m}^{-1}\right)\psi(-\mathrm{tr}X).$$

Since $\sigma(2m)=2m$, both $w_{m,m}$ and $w_{m,m}^{-1}$ have $\varepsilon_{2m}=\begin{pmatrix}0 & \dots & 0 & 1\end{pmatrix}$ as their last row. If the last row of $g\in G$ is $\varepsilon=\varepsilon_m$, then the last row of $\left(\begin{smallmatrix}g & \\ & g\end{smallmatrix}\right)$ is ε_{2m}. Therefore if $\varepsilon g=\varepsilon$, then for any $X\in M$, the matrix $w_{m,m}\left(\begin{smallmatrix}I_m & X\\ & I_m\end{smallmatrix}\right)\left(\begin{smallmatrix}g & \\ & g\end{smallmatrix}\right)w_{m,m}^{-1}$ has ε_{2m} as its last row. Suppose that $w_{m,m}\left(\begin{smallmatrix}I_m & X\\ & I_m\end{smallmatrix}\right)\left(\begin{smallmatrix}g & \\ & g\end{smallmatrix}\right)w_{m,m}^{-1}\in\mathrm{supp}\mathcal{B}_{\pi,\psi}$, then by Corollary 2.3,

$$u_1 w_{m,m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}g & \\ & g\end{pmatrix}w_{m,m}^{-1}u_2=\begin{pmatrix} & & & \lambda_1 I_{n_1}\\ & & \lambda_2 I_{n_2} & \\ & \iddots & & \\ \lambda_r I_{n_r} & & & \end{pmatrix},$$

for $u_1,u_2\in N_{2m}$ and $\lambda_1,\dots,\lambda_r\in F^*$ and $n_1,\dots,n_r$ such that $n_1+\dots+n_r=2m$. Since $u_1,u_2\in N_{2m}$, the last row of u_1,u_2 is ε_{2m} and therefore the product on the left hand side still has ε_{2m} as its last row. This implies $n_r=2m$, $r=1$ and $\lambda_1=1$ and therefore $w_{m,m}\left(\begin{smallmatrix}I_m & X\\ & I_m\end{smallmatrix}\right)\left(\begin{smallmatrix}g & \\ & g\end{smallmatrix}\right)w_{m,m}^{-1}\in N_{2m}$. Therefore $u=w_{m,m}\left(\begin{smallmatrix}g & Xg\\ & g\end{smallmatrix}\right)w_{m,m}^{-1}$ is an upper triangular unipotent matrix. Denote $\left(\begin{smallmatrix}g & Xg\\ & g\end{smallmatrix}\right)=(a_{ij})_{ij}$. Then $u_{ij}=\left(a_{\sigma^{-1}(i),\sigma^{-1}(j)}\right)_{ij}$. For $1\leq j<i\leq m$ we have $\sigma(j)=2j-1<2i-1=\sigma(i)$, and therefore u has 0 in its $(2i-1,2j-1)$ position, and therefore $a_{ij}=g_{ij}=0$. $u_{ii}=1$, for every i, and therefore $a_{ii}=1$ for every i and $g_{ii}=1$ for $1\leq i\leq m$. Therefore g is an upper triangular unipotent matrix, i.e. $g\in N$. For $1\leq j<i\leq m$ we have that $(\sigma(i),\sigma(j+m))=(2i-1,2j)$ and since $j+1\leq i$, this implies $2j<2j+1\leq 2i-1$, and therefore $u_{2i-1,2j}=0$, which implies $a_{i,j+m}=0$. Thus Xg is an upper triangular matrix. Therefore X is an upper triangular matrix.

Therefore the sum

$$J_{\pi,\psi}(W,\phi)=\sum_{\substack{g\in{}_N\backslash^G\\ \varepsilon g=\varepsilon}}\sum_{X\in{}_{\mathcal{B}}\backslash^M}\mathcal{B}_{\pi,\psi}\left(w_{m,m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}g & \\ & g\end{pmatrix}w_{m,m}^{-1}\right)\psi(-\mathrm{tr}X)$$

runs over exactly one coset of ${}_N\backslash^G$ (the coset of I_m) and one coset of ${}_{\mathcal{B}}\backslash^M$ (the coset of 0), and we get that $J_{\pi,\psi}(W,\phi)=\mathcal{B}_{\pi,\psi}(I_{2m})=1\neq 0$. ∎

2.3. **The functional equation.** In this subsection we discuss the functional equation satisfied by the Jacquet-Shalika integrals over a finite field.

Definition 2.5. Let (π, V_π) be a representation of $\mathrm{GL}_{2m}(F)$. We call a vector $v \in V_\pi$ a Shalika vector of π, if $\pi\left(\left(\begin{smallmatrix} g & X \\ & g \end{smallmatrix}\right)\right) v = \Psi\left(\left(\begin{smallmatrix} g & X \\ & g \end{smallmatrix}\right)\right) v$, for every $\left(\begin{smallmatrix} g & X \\ & g \end{smallmatrix}\right) \in S_{2m}$.

Theorem 2.6. *Let (π, V_π) be an irreducible cuspidal representation of $\mathrm{GL}_{2m}(F)$ and suppose that there doesn't exist a non-zero Shalika vector for π. Then there exists a constant $\gamma_{\pi,\psi} \in \mathbb{C}^*$, such that*

$$\tilde{J}_{\pi,\psi}(W, \phi) = \gamma_{\pi,\psi} \cdot J_{\pi,\psi}(W, \phi),$$

for every $\phi : F^m \to \mathbb{C}$ and $W \in \mathcal{W}(\pi, \psi)$.

This will be proved for a p-adic field in Subsection 3.5. The proof is similar for a finite field.

We give an overview of the proof and elaborate on some parts. Note that we assume that F is a finite field.

The idea of the proof is to show that the space of bilinear forms $B : V_\pi \times \mathcal{S}(F^m) \to \mathbb{C}$ which are Ψ-equivariant is at most one dimensional. Since $J_{\pi,\psi}$, $\tilde{J}_{\pi,\psi}$ are non-zero elements of this space, it implies that such a constant exists.

In order to prove that the following space $(\mathrm{Hom}_{S_{2m}}(\pi \otimes \mathcal{S}(F^m), \Psi))$ is at most one-dimensional, we first prove the following multiplicity one theorem (Theorem 3.37):

Theorem 2.7. *Let (π, V_π) be an irreducible cuspidal representation of $\mathrm{GL}_{2m}(F)$, then*

$$\dim \mathrm{Hom}_{P_{2m} \cap M_{m,m}}(\pi, 1) \leq 1.$$

Here P_{2m} is the mirabolic subgroup of $\mathrm{GL}_{2m}(F)$.

Another proof of this theorem for the case that F is a finite field (as in this section) can be found in [Mos08, Theorem 6.1.2]. We give here a brief overview of the proof that will be given in Subsection 3.5.

In order to prove this theorem, we need some preparations. Let n be a positive integer. Suppose that $p \geq q \geq 0$ and $p + q = n$. Let

$$\sigma_{p,q} = \begin{pmatrix} 1 & 2 & \dots & p-q & p-q+1 & p-q+2 & \dots & p & p+1 & p+2 & \dots & p+q \\ 1 & 2 & \dots & p-q & p-q+1 & p-q+3 & \dots & p+q-1 & p-q+2 & p-q+4 & \dots & p+q \end{pmatrix},$$

and let $w_{p,q}$ be the column permutation matrix of $\sigma_{p,q}$. We introduce the following subgroups of $\mathrm{GL}_n(F)$:

$$M_{p,q}^{(n)} = \left\{ \begin{pmatrix} g_p & \\ & g_q \end{pmatrix} \mid g_p \in \mathrm{GL}_p(F),\, g_q \in \mathrm{GL}_q(F) \right\} \qquad p \geq q \geq 0$$

$$M_{p,q-1}^{(n)} = \left\{ \begin{pmatrix} g_p & & \\ & g_{q-1} & \\ & & 1 \end{pmatrix} \mid g_p \in \mathrm{GL}_p(F),\, g_{q-1} \in \mathrm{GL}_{q-1}(F) \right\} \qquad p \geq q \geq 1$$

and we denote

$$\begin{aligned} H_{p,q}^{(n)} &= w_{p,q} M_{p,q}^{(n)} w_{p,q}^{-1}, \\ H_{p,q-1}^{(n-1)} &= w_{p,q} M_{p,q-1}^{(n-1)} w_{p,q}^{-1}, \\ H_{p-1,q-1}^{(n)} &= \left\{ \begin{pmatrix} h & \\ & I_2 \end{pmatrix} \mid h \in H_{p-1,q-1}^{(n-2)} \right\}. \end{aligned}$$

$H_{p,q-1}^{(n)}$ and $H_{p-1,q-1}^{(n-2)}$ can be thought of as subgroups of $\mathrm{GL}_{n-1}(F)$ and $\mathrm{GL}_{n-2}(F)$ respectively. For a positive integer k, we denote by P_k the mirabolic subgroup of $\mathrm{GL}_k(F)$. We denote

$$U_k = \left\{ \begin{pmatrix} I_{k-1} & v \\ & 1 \end{pmatrix} \mid v \in F^{k-1} \right\}.$$

We define for a representation σ of P_{k-1}, a representation σ' of $P_{k-1}U_k$ by $\sigma'(pu) = \psi(u)\sigma(p)$ ($p \in P_{k-1}$, $u \in U_k$) and we define a representation $\Phi^+(\sigma)$ of P_k by $\Phi^+(\sigma) = \mathrm{ind}_{P_{k-1}U_k}^{P_k}(\sigma')$.

We prove the following propositions:

Proposition 2.8. *Suppose $p \geq q \geq 1$ with $p+q=n$. Let (σ, V) be a representation of P_{n-1}. Then there exists an embedding*

$$\mathrm{Hom}_{P_n \cap H_{p,q}^{(n)}}\left(\Phi^+(\sigma), 1\right) \hookrightarrow \mathrm{Hom}_{P_{n-1} \cap H_{p,q-1}^{(n)}}(\sigma, 1).$$

Proposition 2.9. *Suppose $p \geq q \geq 2$ with $p+q=n$. Let (σ, V) be a representation of P_{n-2}. Then there exists an embedding*

$$\mathrm{Hom}_{P_{n-1} \cap H_{p,q-1}^{(n)}}\left(\Phi^+(\sigma), 1\right) \hookrightarrow \mathrm{Hom}_{P_{n-2} \cap H_{p-1,q-1}^{(n)}}(\sigma, 1).$$

The proof of Theorem 2.7 follows by using these propositions repeatedly, the fact that for an irreducible cuspidal representation π of $\mathrm{GL}_n(F)$, its restriction to the mirabolic group P_n is isomorphic to the representation $(\Phi^+)^{n-1}(1)$ ([Gel70, Theorem 2.3]), and by the fact that $P_{2m} \cap H_{m,m} = w_{m,m}(P_{2m} \cap M_{m,m}) w_{m,m}^{-1}$.

Next we construct an embedding $\Lambda : \mathrm{Hom}_{S_{2m} \cap P_{2m}}(\pi, \Psi) \to \mathrm{Hom}_{P_{2m} \cap M_{m,m}}(\pi, 1)$ by the averaging method (Proposition 3.51):

$$\Lambda(L)(v) = \frac{1}{|\mathrm{GL}_m(F)|} \sum_{g \in \mathrm{GL}_m(F)} L\left(\pi\left(\begin{pmatrix} g & \\ & I_m \end{pmatrix}\right) v\right).$$

Unlike the case of a p-adic field, in the case of a finite field there are no convergence issues with this sum. In order to show that Λ is injective, we use the Fourier transform: let $0 \neq L \in \mathrm{Hom}_{S_{2m} \cap P_{2m}}(\pi, \Psi)$ and $v_0 \in V_\pi$ such that $L(v_0) \neq 0$. We define for a function $\eta \in \mathcal{S}(M_m(F))$,

$$v_\eta = \frac{1}{|M_m(F)|} \sum_{X \in M_m(F)} \eta(X)\, \pi\left(\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\right) v_0.$$

A simple computation shows that

$$\Lambda(L)(v_\eta) = \frac{1}{|\mathrm{GL}_m(F)|} \sum_{g \in \mathrm{GL}_m(F)} L\left(\pi\left(\begin{pmatrix} g & \\ & I_m \end{pmatrix}\right) v_0\right) \hat{\eta}(g).$$

By choosing η such that $\hat{\eta} = \delta_{I_m}$ we get that $\Lambda(L)(v_\eta) = \frac{1}{|\mathrm{GL}_m(F)|} L(v) \neq 0$. Therefore we get the following corollary:

Corollary 2.10. *Let (π, V_π) be an irreducible cuspidal representation of $\mathrm{GL}_{2m}(F)$, then*

$$\dim \mathrm{Hom}_{P_{2m} \cap S_{2m}}(\pi, \Psi) \leq 1.$$

We now move to the proof of Theorem 2.6:

Proof. We show that $\dim \mathrm{Hom}_{S_{2m}}(\pi \otimes \mathcal{S}(F^m), \Psi) \leq 1$. Since $J_{\pi,\psi}, \tilde{J}_{\pi,\psi} \in \mathrm{Hom}_{S_{2m}}(\pi \otimes \mathcal{S}(F^m), \Psi)$, and both are non-zero forms, this will imply that there exists such constant.

We first consider the restriction map

$$\begin{aligned} \mathrm{Hom}_{S_{2m}}(\pi \otimes \mathcal{S}(F^m), \Psi) &\to \mathrm{Hom}_{S_{2m}}(\pi \otimes \mathcal{S}(F^m \setminus \{0\}), \Psi) \\ B &\mapsto B \restriction_{V_\pi \times \mathcal{S}(F^m \setminus \{0\})}. \end{aligned}$$

This map is injective. Indeed, suppose that $B : V_\pi \times \mathcal{S}(F^m) \to \mathbb{C}$ such that $B \restriction_{V_\pi \times \mathcal{S}(F^m \setminus \{0\})} \equiv 0$ and $B \neq 0$. Then the map $\beta : V_\pi \to \mathbb{C}$ defined as $\beta(v) = B(v, \delta_0)$ is a non-zero linear functional. Let $(\cdot, \cdot)$ be an inner product on V_π, with respect to which π is unitary. Then there exists a non-zero vector v_0 such that $\beta(v) = (v, v_0)$, for every $v \in V_\pi$. Let $v \in V_\pi$ and $\left(\begin{smallmatrix} g & X \\ & g \end{smallmatrix}\right) \in S_{2m}$. From the equivariance properties of B, and since $\rho(g)\delta_0 = \delta_0$, we have that

$$\beta\left(\pi\left(\begin{pmatrix} g & X \\ & g \end{pmatrix}\right) v\right) = \Psi\left(\begin{pmatrix} g & X \\ & g \end{pmatrix}\right) \beta(v),$$

which implies $\pi\left(\left(\begin{smallmatrix} g & X \\ & g \end{smallmatrix}\right)\right) v_0 = \Psi\left(\left(\begin{smallmatrix} g & X \\ & g \end{smallmatrix}\right)\right) v_0$, i.e. $v_0 \neq 0$ is a Shalika vector, which contradicts our assumption.

We now write

$$\begin{aligned} \mathrm{Hom}_{S_{2m}}(\pi \otimes \mathcal{S}(F^m \setminus \{0\}), \Psi) &= \mathrm{Hom}_{S_{2m}}\left(\left(\Psi^{-1}\pi\right) \otimes \mathcal{S}(F^m \setminus \{0\}), 1\right) \\ &\cong \mathrm{Hom}_{S_{2m}}\left(\Psi^{-1}\pi, \widetilde{\mathcal{S}(F^m \setminus \{0\})}\right). \end{aligned}$$

We identify $F^m \setminus \{0\}$ with ${}_{S_{2m} \cap P_{2m}} \backslash^{S_{2m}}$ using the map $\left(\begin{smallmatrix} g & X \\ & g \end{smallmatrix}\right) \mapsto \varepsilon_m g$ and therefore $\mathcal{S}(F^m \setminus \{0\}) \cong \mathcal{S}\left({}_{S_{2m} \cap P_{2m}} \backslash^{S_{2m}}\right) = \mathrm{ind}_{S_{2m} \cap P_{2m}}^{S_{2m}}(1)$.

$$\begin{aligned} \mathrm{Hom}_{S_{2m}}(\pi \otimes \mathcal{S}(F^m \setminus \{0\}), \Psi) &\cong \mathrm{Hom}_{S_{2m}}\left(\Psi^{-1}\pi, \widetilde{\mathrm{ind}_{S_{2m} \cap P_{2m}}^{S_{2m}}(1)}\right) \\ &\cong \mathrm{Hom}_{S_{2m}}\left(\Psi^{-1}\pi, \mathrm{ind}_{S_{2m} \cap P_{2m}}^{S_{2m}}(\tilde{1})\right). \end{aligned}$$

By Frobenius reciprocity

$$\begin{aligned} \mathrm{Hom}_{S_{2m}}\left(\Psi^{-1}\pi, \mathrm{ind}_{S_{2m} \cap P_{2m}}^{S_{2m}}(1)\right) &\cong \mathrm{Hom}_{P_m \cap S_{2m}}\left(\Psi^{-1}\pi \restriction_{P_m \cap S_{2m}}, 1\right) \\ &= \mathrm{Hom}_{P_m \cap S_{2m}}(\pi, \Psi) \end{aligned}$$

By Corollary 2.10, we have $\dim \mathrm{Hom}_{P_m \cap S_{2m}}(\pi, \Psi) \leq 1$, and the theorem is proved. ☐

Remark 2.11. As seen in the proof, this proof fails when π admits a Shalika vector. In this case, a modified functional equation is valid. This is discussed in Subsection 4.4.

2.3.1. *Equivalent conditions for the existence of a Shalika vector.* Let (π, V_π) be an irreducible cuspidal representation of $\mathrm{GL}_{2m}(\mathbb{F}_q)$, and denote $G = \mathrm{GL}_m(\mathbb{F}_q)$. There exists a regular character $\theta : \mathbb{F}_{q^{2m}}^* \to \mathbb{C}^*$ which is associated to π [Gre55]. We present an equivalent criterion for π to admit a Shalika vector, in terms of θ.

We denote

$$V_{\pi_{N_{m,m},\psi}} = \left\{ v \in V_\pi \mid \pi\left(\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\right) v = \psi(\mathrm{tr}X)\, v,\ \forall X \in M_m(\mathbb{F}_q) \right\},$$

a twisted Jacquet module. This space is invariant under the action $\pi\left(\left(\begin{smallmatrix} g & \\ & g \end{smallmatrix}\right)\right)$ for $g \in G$. We denote its action by $\pi_{N_{m,m},\psi}(g) = \pi\left(\left(\begin{smallmatrix} g & \\ & g \end{smallmatrix}\right)\right) \restriction_{V_{\pi_{N_{m,m},\psi}}}$.

A non-zero Shalika vector v is an element $0 \neq v \in V_{\pi_{N_{m,m},\psi}}$, such that $\pi\left(\left(\begin{smallmatrix} g & \\ & g \end{smallmatrix}\right)\right) v = v$ for every $g \in G$, and therefore it exists if and only if $\mathrm{Hom}_G\left(1, \pi_{N_{m,m},\psi}\right) \neq 0$.

Due to a result of Prasad [Pra00, Theorem 1], $\pi_{N_{m,m},\psi} \cong \mathrm{Ind}_{\mathbb{F}_{q^m}^*}^{G}\left(\theta \restriction_{\mathbb{F}_{q^m}^*}\right)$ (we view $\mathbb{F}_{q^m}^*$ as a subgroup of $\mathrm{GL}_m(\mathbb{F}_q)$). Therefore π admits a Shalika vector if and only if

$$0 \neq \mathrm{Hom}_G\left(1, \mathrm{Ind}_{\mathbb{F}_{q^m}^*}^{G}\left(\theta \restriction_{\mathbb{F}_{q^m}^*}\right)\right).$$

By Frobenius reciprocity

$$\mathrm{Hom}_G\left(1, \mathrm{Ind}_{\mathbb{F}_{q^m}^*}^{G}\left(\theta \restriction_{\mathbb{F}_{q^m}^*}\right)\right) \cong \mathrm{Hom}_{\mathbb{F}_{q^m}^*}\left(1 \restriction_{\mathbb{F}_{q^m}^*}, \theta \restriction_{\mathbb{F}_{q^m}^*}\right)$$

and the last space is non zero if and only if $\theta \restriction_{\mathbb{F}_{q^m}^*} \equiv 1$, and then it is one dimensional.

Corollary 2.12. *Let (π, V_π) be an irreducible cuspidal representation of $\mathrm{GL}_{2m}(\mathbb{F}_q)$ and let $\theta : \mathbb{F}_{q^{2m}}^* \to \mathbb{C}^*$ be a regular character associated with π. Then π admits a non-zero Shalika vector if and only if $\theta \restriction_{\mathbb{F}_{q^m}^*} \equiv 1$. In this case, the space of Shalika vectors is one dimensional.*

We finish by giving another criterion for admitting a non-zero Shalika vector.

Proposition 2.13. *Let (π, V_π) be an irreducible cuspidal representation of $\mathrm{GL}_{2m}(\mathbb{F}_q)$. π admits a non-zero Shalika vector if and only if there exists $W \in \mathcal{W}(\pi, \psi)$ such that*

$$J_{\pi,\psi}(W, 1) \neq 0.$$

Proof. Suppose that there exists $W \in \mathcal{W}(\pi, \psi)$ such that

$$\sum_{g \in N \backslash G} \sum_{X \in \mathcal{B} \backslash M} W\left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix}\right) \psi(-\mathrm{tr}X) \neq 0.$$

Denote

$$W_0(g) = \sum_{k \in N \backslash G} \sum_{X \in \mathcal{B} \backslash M} W\left(g \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} k & \\ & k \end{pmatrix}\right) \psi(-\mathrm{tr}X).$$

Then $W_0 \in \mathcal{W}(\pi, \psi)$ as a linear combination of right translations of W. $W_0 \neq 0$ as $W_0(w_{m,m}) \neq 0$. Clearly, W_0 is a non-zero Shalika vector.

We now move to prove the other direction. Assume that π admits a non-zero Shalika vector v_0. This vector defines a non-zero element $T_0 \in \mathrm{Hom}_{S_{2m}}(\pi, \Psi)$ by $T_0(v) = (v, v_0)$, where $(\cdot,\cdot)$ is an inner product with respect to which π is unitary. Since $\mathrm{Hom}_{S_{2m}}(\pi,\Psi)\subseteq \mathrm{Hom}_{P_{2m}\cap S_{2m}}(\pi,\Psi)$, we have $\mathrm{Hom}_{P_{2m}\cap S_{2m}}(\pi,\Psi)\neq 0$. Due to Corollary 2.10,

$$\dim \mathrm{Hom}_{P_{2m}\cap S_{2m}}(\pi,\Psi)\leq 1,$$

and therefore we have in this case (that π admits a non-zero Shalika vector) that $\mathrm{Hom}_{P_{2m}\cap S_{2m}}(\pi,\Psi) = \mathrm{Hom}_{S_{2m}}(\pi,\Psi)$.

We present a non-zero element of $\mathrm{Hom}_{P_{2m}\cap S_{2m}}(\pi,\Psi)$ defined by

$$W(g)=\sum_{k\in N\backslash P}\sum_{X\in\mathcal{B}\backslash M}\mathcal{B}_{\pi,\psi}\left(g\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}k & \\ & k\end{pmatrix}w_{m,m}^{-1}\right)\psi(-\mathrm{tr}X),$$

where $P=P_m(\mathbb{F}_q)=\{g\in \mathrm{GL}_m(\mathbb{F}_q)\mid \varepsilon_m g=\varepsilon_m\}$. As above, it is clear that $W\in \mathrm{Hom}_{P_{2m}\cap S_{2m}}(\pi,\Psi)$. From Proposition 2.4, $W(w_{m,m})=1$ and therefore $W\neq 0$. Since $\mathrm{Hom}_{P_{2m}\cap S_{2m}}(\pi,\Psi) = \mathrm{Hom}_{S_{2m}}(\pi,\Psi)$, we have $W\in \mathrm{Hom}_{S_{2m}}(\pi,\Psi)$. A direct computation shows that

$$J_{\pi,\psi}(W,1)=W(w_{m,m})\neq 0.$$

$\square$

2.4. **Computations.** We now compute $\gamma_{\pi,\psi}$ for cuspidal representations of $\mathrm{GL}_{2m}(\mathbb{F}_q)$ that don't admit a Shalika vector, where $m=1,2$. We begin with a general computation.

Let $f:\mathbb{F}_q^m\to\mathbb{C}$ be defined as

$$f(x)=\delta_{-\varepsilon_1}(x)=\begin{cases}1 & x=-\varepsilon_1=(-1,0,\dots,0)\\ 0 & x\neq-\varepsilon_1\end{cases}.$$

Then

$$\hat{f}(y)=\frac{1}{\left|\mathbb{F}_q^m\right|}\sum_{a\in\mathbb{F}_q^m}f(a)\,\psi^{\mathcal{F}}(\langle a,y\rangle)=\frac{1}{q^m}\psi^{\mathcal{F}}(-y_1),$$

and by Fourier inversion formula, $\hat{\hat{f}}(x)=\frac{1}{q^m}f(-x)$, and therefore if $h(x)=\psi^{\mathcal{F}}(-x_1)$, then $\hat{h}(x)=\delta_{\varepsilon_1}(x)$.

We substitute $\phi(x)=\psi^{\mathcal{F}}(-x_1)$ ($\hat{\phi}(x)=\delta_{\varepsilon_1}(x)$) and $W(g)=[G:N]\,[M:\mathcal{B}]\,\mathcal{B}_{\pi,\psi}\left(gw_{m,m}^{-1}\right)$ in the equality

$$\tilde{J}_{\pi,\psi}(W,\phi)=\gamma_{\pi,\psi}\cdot J_{\pi,\psi}(W,\phi),$$

in order to compute $\gamma_{\pi,\psi}$.

We begin with computing

$$\tilde{J}_{\pi,\psi}(W,\phi)=\sum_{g\in N\backslash G}\sum_{X\in\mathcal{B}\backslash M}\mathcal{B}_{\pi,\psi}\left(w_{m,m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}g & \\ & g\end{pmatrix}w_{m,m}^{-1}\right)\psi(-\mathrm{tr}X)\cdot\delta_{\varepsilon_1}\left(\varepsilon_1 g^{l}\right).$$

$\delta_{\varepsilon_1}\left(\varepsilon_1 g^l\right)$ equals 1 if and only if the first row of g^l equals $\varepsilon_1=\begin{pmatrix}1 & 0 & \dots & 0\end{pmatrix}$. This is true if and only if the first column of g^{-1} is ε_1^t. By matrix multiplication we see that this is true if and only if the first column of g is ε_1^t.

Suppose that $g \in G$ such that g has ε_1^t as its first column. We recall that by Corollary 2.3 that if $w_{m,m} \left(\begin{smallmatrix} I_m & X \\ & I_m \end{smallmatrix}\right) \left(\begin{smallmatrix} g & \\ & g \end{smallmatrix}\right) w_{m,m}^{-1} \in \operatorname{supp}\mathcal{B}_{\pi,\psi}$, then

$$u_1 w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} w_{m,m}^{-1} u_2 = \begin{pmatrix} & & & \lambda_1 I_{n_1} \\ & & \lambda_2 I_{n_2} & \\ & \iddots & & \\ \lambda_r I_{n_r} & & & \end{pmatrix},$$

for $u_1, u_2 \in N_{2m}$ and $\lambda_1, \dots, \lambda_r \in F^*$ and $n_1, \dots, n_r$, such that $n_1 + \dots + n_r = 2m$. Since g has ε_1^t as its first column, $\left(\begin{smallmatrix} g & \\ & g \end{smallmatrix}\right)$ has $\varepsilon_1^t \in F^{2m \times 1}$ as its first column. Since $\sigma(1) = 1$, the elements $w_{m,m}$, $w_{m,m}^{-1}$ have $\varepsilon_1^t \in F^{2m \times 1}$ as their first column, and since u_1, u_2 are upper triangular unipotent elements, they also have $\varepsilon_1^t \in F^{2m \times 1}$ as their first column. Therefore the left hand side has ε_1^t as its first column, and therefore $r = 1$, $\lambda_1 = 1$ and $n_1 = 2m$, and we have $w_{m,m} \left(\begin{smallmatrix} I_m & X \\ & I_m \end{smallmatrix}\right) \left(\begin{smallmatrix} g & \\ & g \end{smallmatrix}\right) w_{m,m}^{-1} \in N_{2m}$. As in the proof of Proposition 2.4, this implies that $g \in N$, $X \in \mathcal{B}$, and therefore

$$\tilde{J}_{\pi,\psi}(W, \phi) = \mathcal{B}_{\pi,\psi}(I_{2m}) = 1.$$

Therefore

$$\gamma_{\pi,\psi}^{-1} = J_{\pi,\psi}(W,\phi) = \sum_{g \in N \backslash G} \sum_{X \in \mathcal{B} \backslash M} \mathcal{B}_{\pi,\psi}\left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} w_{m,m}^{-1} \right) \psi(-\mathrm{tr} X) \cdot \psi^{\mathcal{F}}(-g_{m1}).$$

We denote for $a \in \mathbb{F}_q$,

$$S_a = \sum_{\substack{g \in N \backslash G \\ g_{m1} = a}} \sum_{X \in \mathcal{B} \backslash M} \mathcal{B}_{\pi,\psi}\left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} w_{m,m}^{-1} \right) \psi(-\mathrm{tr} X).$$

Then $\gamma_{\pi,\psi}^{-1} = \sum_{a \in \mathbb{F}_q} S_a \psi^{\mathcal{F}}(-a)$. For $a \neq 0$, replacing g with ag in the expression of S_a yields $S_a = \omega_\pi(a) S_1$. Therefore $\gamma_{\pi,\psi}^{-1} = S_0 + S_1 \sum_{a \in \mathbb{F}_q^*} \psi^{\mathcal{F}}(-a) \omega_\pi(a)$.

Note that if the central character ω_π is not trivial, then $\omega_\pi(a) \neq 0$ for some $a \in \mathbb{F}_q^*$, and then by replacing g with ag in S_0 we get $S_0 = \omega_\pi(a) S_0$, and therefore $S_0 = 0$.

Regarding S_1, we define for $v \in \mathbb{F}_q^{m-1}$,

$$S_{(1,v)} = \sum_{\substack{g \in N \backslash G \\ \varepsilon_m g = (1,v)}} \sum_{X \in \mathcal{B} \backslash M} \mathcal{B}_{\pi,\psi}\left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} w_{m,m}^{-1} \right) \psi(-\mathrm{tr} X),$$

and therefore $S_1 = \sum_{v \in \mathbb{F}_q^{m-1}} S_{(1,v)}$. For $v \in \mathbb{F}_q^{m-1}$, denote $u_v = \left(\begin{smallmatrix} 1 & v \\ & I_{m-1} \end{smallmatrix}\right)$, then $\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} u_v = \begin{pmatrix} 1 & v \end{pmatrix}$, and therefore $\varepsilon_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 1 & v \end{pmatrix} u_v^{-1}$. Substituting $g = g' u_v$ in $S_{(1,v)}$ yields

$$S_{(1,v)} = \sum_{\substack{g' \in N \backslash G \\ \varepsilon_m g' = \varepsilon_1}} \sum_{X \in \mathcal{B} \backslash M} \mathcal{B}_{\pi,\psi}\left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g' & \\ & g' \end{pmatrix} \begin{pmatrix} u_v & \\ & u_v \end{pmatrix} w_{m,m}^{-1} \right) \psi(-\mathrm{tr} X).$$

We now compute $w_{m,m} \left(\begin{smallmatrix} u_v & \\ & u_v \end{smallmatrix}\right) w_{m,m}^{-1}$: its diagonal consists of the element 1 only. The only possible non-diagonal non-zero elements of u_v are those with index $(1, j)$ and $(m+1, m+j)$ with $1 < j \leq m$. These move after conjugation to $(\sigma(1), \sigma(j)) = (1, 2j-1)$ and $(\sigma(m+1), \sigma(m+j)) =$

$(2,2j)$. Therefore $w_{m,m}\left(\begin{smallmatrix}u_v & \\ & u_v\end{smallmatrix}\right)w_{m,m}^{-1}$ is an upper triangular unipotent matrix, with no non-zero elements above its diagonal, and therefore $\psi\left(w_{m,m}\left(\begin{smallmatrix}u_v & \\ & u_v\end{smallmatrix}\right)w_{m,m}^{-1}\right)=1$. Hence

$$\mathcal{B}_{\pi,\psi}\left(w_{m,m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}g' & \\ & g'\end{pmatrix}\begin{pmatrix}u_v & \\ & u_v\end{pmatrix}w_{m,m}^{-1}\right)=\mathcal{B}_{\pi,\psi}\left(w_{m,m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}g' & \\ & g'\end{pmatrix}w_{m,m}^{-1}\right).$$

Therefore we have $S_{(1,v)}=S_{\varepsilon_1}$, and $S_1=q^{m-1}S_{\varepsilon_1}$ and $\gamma_{\pi,\psi}^{-1}=S_0+q^{m-1}\left(\sum_{a\in\mathbb{F}_q^*}\omega_\pi(a)\,\psi^{\mathcal{F}}(-a)\right)S_{\varepsilon_1}$.

2.4.1. *Computation for $m=1$.* For $m=1$, $G=\mathrm{GL}_1(\mathbb{F}_q)=\mathbb{F}_q^*$ and $M=M_1(\mathbb{F}_q)=\mathbb{F}_q$ and therefore $\mathcal{B}=M$ and $N=\{1\}$ and the condition $\varepsilon g=\varepsilon_1$ implies $g=1$. Therefore

$$S_{\varepsilon_1}=\mathcal{B}_{\pi,\psi}\left(w_{m,m}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}w_{m,m}^{-1}\right)=\mathcal{B}_{\pi,\psi}(I_2)=1,$$

and $S_0=0$ as the condition $g_{11}=0$ implies $g=0$ but then g is not invertible, and hence S_0 is the empty sum. $q^{m-1}=1$ and we have

$$\gamma_{\pi,\psi}^{-1}=\sum_{a\in\mathbb{F}_q^*}\omega_\pi(a)\,\psi^{\mathcal{F}}(-a).$$

We conclude this in a theorem:

Theorem 2.14. *Let π be an irreducible cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_q)$. Then*

$$\gamma_{\pi,\psi}^{-1}=\sum_{a\in\mathbb{F}_q^*}\omega_\pi(a)\,\psi^{\mathcal{F}}(-a).$$

2.4.2. *Computation for $m=2$.* For $m=2$, $G=\mathrm{GL}_2(\mathbb{F}_q)$. Let $\theta:\mathbb{F}_{q^4}^*\to\mathbb{C}$ be a regular character associated with π and assume that $\theta\restriction_{\mathbb{F}_{q^2}^*}\not\equiv 1$, so that π doesn't admit a Shalika vector.

We begin with computing S_0 in the case that the central character is trivial. Let $g\in\mathrm{GL}_2(\mathbb{F}_q)$, such that $g_{21}=0$. Then $g=\left(\begin{smallmatrix}a & c\\ 0 & b\end{smallmatrix}\right)=\left(\begin{smallmatrix}1 & \frac{c}{b}\\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}a & \\ & b\end{smallmatrix}\right)$, and therefore $g\in N\left(\begin{smallmatrix}a & \\ & b\end{smallmatrix}\right)$ for $a,b\in\mathbb{F}_q^*$. Then

$$S_0=\sum_{\substack{a\in\mathbb{F}_q^*\\ b\in\mathbb{F}_q^*}}\sum_{X\in\mathcal{B}\backslash M}\mathcal{B}_{\pi,\psi}\left(w_{m,m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}\mathrm{diag}(a,b) & \\ & \mathrm{diag}(a,b)\end{pmatrix}w_{m,m}^{-1}\right)\psi(-\mathrm{tr}X).$$

Taking bI_4 out of $\mathcal{B}_{\pi,\psi}$, in exchange of multiplying by the central character $\omega_\pi(b)=1$, and then replacing ab^{-1} with a and $\left(\begin{smallmatrix}a & \\ & 1\end{smallmatrix}\right)$ with g we get

$$S_0=\sum_{b\in\mathbb{F}_q^*}\sum_{\substack{g\in N\backslash G\\ \varepsilon g=\varepsilon}}\sum_{X\in\mathcal{B}\backslash M}\mathcal{B}_{\pi,\psi}\left(w_{m,m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}g & \\ & g\end{pmatrix}w_{m,m}^{-1}\right)\psi(-\mathrm{tr}X).$$

By Proposition 2.4, we get $S_0=q-1$. We conclude that $S_0=\begin{cases}0 & \omega_\pi\not\equiv 1\\ q-1 & \omega_\pi\equiv 1\end{cases}$.

We now compute S_{ε_1}. Suppose $g\in\mathrm{GL}_2(\mathbb{F}_q)$ with $\varepsilon_m g=\varepsilon_1$ i.e. $g=\left(\begin{smallmatrix}a & b\\ 1 & 0\end{smallmatrix}\right)$ with $b\in\mathbb{F}_q^*$. Then $g=\left(\begin{smallmatrix}1 & a\\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & b\\ 1 & 0\end{smallmatrix}\right)$, and therefore $g\in N_2(\mathbb{F}_q)\left(\begin{smallmatrix}0 & b\\ 1 & 0\end{smallmatrix}\right)$.

Since

$$_{\mathcal{B}_2(\mathbb{F}_q)}\backslash^{M_2(\mathbb{F}_q)}\cong\mathcal{N}_2^-(\mathbb{F}_q),$$

where $\mathcal{N}_2^-(\mathbb{F}_q)$ is the subspace consisting of lower triangular nilpotent elements of $M_2(\mathbb{F}_q)$, it suffices to consider only these elements.

Let $X = \left(\begin{smallmatrix} 0 & 0 \\ x & 0 \end{smallmatrix}\right)$ and let $g = \left(\begin{smallmatrix} 0 & b \\ 1 & 0 \end{smallmatrix}\right)$ where $b \in \mathbb{F}_q^*$. Then a simple computation shows that

$$w_{m,m} \begin{pmatrix} I & X \\ & I \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} w_{m,m}^{-1} = \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & xb & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$\mathcal{B}_{\pi,\psi}\left(w_{m,m} \begin{pmatrix} I & X \\ & I \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} w_{m,m}^{-1}\right) \psi\left(-\mathrm{tr}\left(X\right)\right) = \mathcal{B}_{\pi,\psi}\left(\begin{pmatrix} 0 & bI_2 \\ I_2 & 0 \end{pmatrix}\right),$$

which implies

$$S_{\varepsilon_1} = \sum_{x \in \mathbb{F}_q} \sum_{b \in \mathbb{F}_q^*} \mathcal{B}_{\pi,\psi}\left(\begin{pmatrix} 0 & bI_2 \\ I_2 & 0 \end{pmatrix}\right) = q \sum_{b \in \mathbb{F}_q^*} \mathcal{B}_{\pi,\psi}\left(\begin{pmatrix} 0 & bI_2 \\ I_2 & 0 \end{pmatrix}\right).$$

We use the values of the Bessel function for $\mathrm{GL}_4(\mathbb{F}_q)$, which are computed by Deriziotis and Gotsis [DG98, Page 103]. In our case

$$w = w_6 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad t = \begin{pmatrix} \mu I_2 & 0 \\ 0 & \nu I_2 \end{pmatrix},$$

where $\mu = b$, $\nu = 1$. The value $\mathcal{B}_{\pi,\psi}(tw)$ is given by

$$\mathcal{B}_{\pi,\psi}(tw) = \sum_{\substack{\xi \in \mathbb{F}_{q^4}^* \\ \mathrm{N}_{\mathbb{F}_{q^4}/\mathbb{F}_q}(\xi) = \mu^2\nu^2}} F_6(\xi, t)\, \theta(\xi),$$

where

$$F_6(\xi, t) = -q^{-4}\left(F_6'(\xi, t) + \sum_{\beta \in \mathbb{F}_q^*} \psi\left(-\beta + \frac{a_1(\xi) + a_3(\xi)\mu\nu}{\beta\mu\nu^2}\right)\right),$$

and

$$F_6'(\xi, t) = \begin{cases} -q & \xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \text{ and } \mu\nu = -\mathrm{N}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\xi) \\ 0 & \text{otherwise} \end{cases},$$

$$a_3(\xi) = -\mathrm{Tr}_{\mathbb{F}_{q^4}/\mathbb{F}_q}(\xi) = -\left(\xi + \xi^q + \xi^{q^2} + \xi^{q^3}\right),$$

$$a_1(\xi) = -\left(\xi^{1+q+q^2} + \xi^{1+q+q^3} + \xi^{1+q^2+q^3} + \xi^{q+q^2+q^3}\right),$$

$$\mathrm{N}_{\mathbb{F}_{q^4}/\mathbb{F}_q}(\xi) = \xi^{1+q+q^2+q^3},$$

$$\mathrm{N}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\xi) = \xi^{q+1}.$$

In our case,

$$F_6(\xi, t) = -\,q^{-4}\left(F_6'(\xi, t) + \sum_{\beta \in \mathbb{F}_q^*} \psi\left(-\beta + \frac{a_1(\xi) + a_3(\xi)\, b}{\beta b}\right)\right).$$

Hence

$$\mathcal{B}_{\pi,\psi}(tw) = \sum_{\substack{\xi\in\mathbb{F}_{q^4}^* \\ \mathrm{N}_{\mathbb{F}_{q^4}/\mathbb{F}_q}(\xi)=b^2}} F_6(\xi,t)\,\theta(\xi)$$

$$= -\frac{1}{q^4}\sum_{\substack{\xi\in\mathbb{F}_{q^4}^* \\ \mathrm{N}_{\mathbb{F}_{q^4}/\mathbb{F}_q}(\xi)=b^2}}\sum_{\beta\in\mathbb{F}_q^*}\psi\left(-\beta+\frac{a_1(\xi)-b\mathrm{Tr}_{\mathbb{F}_{q^4}/\mathbb{F}_q}(\xi)}{\beta b}\right)\theta(\xi)+\frac{1}{q^3}\sum_{\substack{\xi\in\mathbb{F}_{q^2}\setminus\mathbb{F}_q \\ \mathrm{N}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\xi)=-b}}\theta(\xi),$$

and therefore

$$q^{m-1}S_{\varepsilon_1} = q^{m-1}\cdot q\sum_{b\in\mathbb{F}_q^*}\mathcal{B}_{\pi,\psi}\left(\begin{pmatrix}0 & bI_2\\ I_2 & 0\end{pmatrix}\right)$$

$$= \sum_{b\in\mathbb{F}_q^*}\left(-\frac{1}{q^2}\sum_{\substack{\xi\in\mathbb{F}_{q^4}^* \\ \mathrm{N}_{\mathbb{F}_{q^4}/\mathbb{F}_q}(\xi)=b^2}}\sum_{\beta\in\mathbb{F}_q^*}\psi\left(-\beta+\frac{a_1(\xi)-b\mathrm{Tr}_{\mathbb{F}_{q^4}/\mathbb{F}_q}(\xi)}{\beta b}\right)\theta(\xi)+\frac{1}{q}\sum_{\substack{\xi\in\mathbb{F}_{q^2}\setminus\mathbb{F}_q \\ \mathrm{N}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\xi)=-b}}\theta(\xi)\right).$$

It is clear that $\sum_{b\in\mathbb{F}_q^*}\sum_{\substack{\xi\in\mathbb{F}_{q^2}\setminus\mathbb{F}_q \\ \mathrm{N}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\xi)=-b}}\theta(\xi) = \sum_{\xi\in\mathbb{F}_{q^2}^*}\theta(\xi)-\sum_{\xi\in\mathbb{F}_q^*}\theta(\xi)$, as $-b$ runs on all the norms of elements of $\mathbb{F}_{q^2}\setminus\mathbb{F}_q$. Since $\theta\restriction_{\mathbb{F}_{q^2}^*}\not\equiv 1$, $\sum_{\xi\in\mathbb{F}_{q^2}^*}\theta(\xi)=0$. Regarding the sum over $\mathbb{F}_q^*$, Green's formulas imply that $\omega_\pi=\theta\restriction_{\mathbb{F}_q^*}$, and therefore we have $\sum_{\xi\in\mathbb{F}_q^*}\theta(\xi) = \begin{cases} q-1 & \omega_\pi\equiv 1\\ 0 & \omega_\pi\not\equiv 1\end{cases}$. We also notice that if $\omega_\pi\equiv 1$, then $\sum_{a\in\mathbb{F}_q^*}\omega_\pi(a)\,\psi^{\mathcal{F}}(-a)=-1$. Combining these implies

$$\gamma_{\pi,\psi}^{-1} = T_0-\frac{1}{q^2}\left(\sum_{a\in\mathbb{F}_q^*}\omega_\pi(a)\,\psi^{\mathcal{F}}(-a)\right)\left(\sum_{b\in\mathbb{F}_q^*}\left(\sum_{\substack{\xi\in\mathbb{F}_{q^4}^* \\ \mathrm{N}_{\mathbb{F}_{q^4}/\mathbb{F}_q}(\xi)=b^2}}\sum_{\beta\in\mathbb{F}_q^*}\psi\left(-\beta+\frac{a_1(\xi)-b\mathrm{Tr}_{\mathbb{F}_{q^4}/\mathbb{F}_q}(\xi)}{\beta b}\right)\theta(\xi)\right)\right),$$

where $T_0 = S_0+\frac{1}{q}(q-1) = \begin{cases} q-\frac{1}{q} & \omega_\pi\equiv 1\\ 0 & \omega_\pi\not\equiv 1\end{cases}$.

Using the relation $a_1(\xi) = -\mathrm{N}_{\mathbb{F}_{q^4}/\mathbb{F}_q}(\xi)\cdot\mathrm{Tr}_{\mathbb{F}_{q^4}/\mathbb{F}_q}\left(\frac{1}{\xi}\right)$, we obtain the following theorem.

Theorem 2.15. *Let π be an irreducible cuspidal representation of $\mathrm{GL}_4(\mathbb{F}_q)$. Then*

$$\gamma_{\pi,\psi}^{-1} = T_0-\frac{1}{q^2}\left(\sum_{a\in\mathbb{F}_q^*}\omega_\pi(a)\,\psi^{\mathcal{F}}(-a)\right)\left(\sum_{b\in\mathbb{F}_q^*}\left(\sum_{\substack{\xi\in\mathbb{F}_{q^4}^* \\ \mathrm{N}_{\mathbb{F}_{q^4}/\mathbb{F}_q}(\xi)=b^2}}\sum_{\beta\in\mathbb{F}_q^*}\psi^{-1}\left(\beta+\frac{1}{\beta}\mathrm{Tr}_{\mathbb{F}_{q^4}/\mathbb{F}_q}\left(\xi+\frac{b}{\xi}\right)\right)\theta(\xi)\right)\right),$$

$$where\ T_0 = \begin{cases} q - \frac{1}{q} & \omega_\pi \equiv 1 \\ 0 & \omega_\pi \not\equiv 1 \end{cases}.$$

3. The Jacquet-Shalika integral over a p-adic field

In this section, F is a p-adic field. We denote by $\mathcal{O}$ the ring of integers of F, $\mathcal{P}$ the unique prime ideal of $\mathcal{O}$, and ϖ a uniformizer of F (a generator of $\mathcal{P}$). We denote $q = \left|{}^{\mathcal{O}}/_{\mathcal{P}}\right|$.

3.1. **Preliminaries.**

3.1.1. *Decomposition of Haar measures.* Let G be an l-group. It is common knowledge that there exists a unique (up to multiplication by a positive scalar) measure which is right invariant to the action of G, i.e. there exists a measure μ_G such that

$$\int_G f(ga)\, d\mu_{r,G}(g) = \int_G f(g)\, d\mu_{r,G}(g),$$

for every $f \in \mathcal{S}(G)$, $a \in G$. A similar result holds for a left invariant Haar measure.

We will need some decomposition theorems.

Theorem 3.1. *Let G be a locally compact unimodular group, and let $P, K \leq G$ be two closed subgroups of G, such that $G = PK$ and such that $P \cap K$ is compact. Then a Haar measure on G is given by $\int_K \int_P f(pk)\, d\mu_{l,P} d\mu_{r,K}$ where $d\mu_{l,P}$ is a left Haar measure on P and $d\mu_{r,K}$ is a right Haar measure on K.*

Theorem 3.2. *Let B be a locally compact group, and suppose that $B = A \ltimes N$ where A, N are closed subgroups of B. Then a left Haar measure on B is given by $\int_A \int_N f(an)\, d\mu_{l,N}(n)\, d\mu_{l,A}(a)$ where $\mu_{l,A}, \mu_{l,N}$ are left Haar measures corresponding to A, N.*

Another form for a left Haar measure on B is given by $\int_A \delta_B^{-1}(a) \int_N f(na)\, d\mu_{l,N}(n)\, d\mu_{l,A}(a)$ where δ_B is the Haar modular function of the group B, i.e: $\int_B f(gb)\, d\mu_{l,B}(g) = \delta_B(b) \int_B f(g)\, d\mu_{l,B}(g)$ ($b \in B$).

3.1.2. *Iwasawa decomposition.* Let n be a positive integer. Denote $G = \mathrm{GL}_n(F)$, $K = \mathrm{GL}_n(\mathcal{O})$ and denote by B the Borel subgroup of G, consisting of invertible upper-triangular matrices. B is a closed subgroup of G.

The Iwasawa decomposition of G is given by $G = BK$.

It is standard knowledge that G is unimodular. K is also unimodular as a compact group.

Since $B \cap K$ is compact, we get the following decomposition of the Haar measure (using Theorem 3.1): Given a function $f \in C^\infty(G)$ (i.e. a smooth function $f : G \to \mathbb{C}$) we have

$$\int_G f(g)\, d\mu_G(g) = \int_B \int_K f(bk)\, d\mu_K(k)\, d\mu_B(b).$$

We denote by $A \subseteq G$ the diagonal matrix subgroup of G and by N the upper triangular unipotent matrix subgroup of G. It is clear that $B = A \ltimes N$. N, A are unimodular. We write the decomposition of the Haar measure on B as well (using Theorem 3.2):

$$\int_B f(b)\, d\mu_B(b) = \int_A \delta_B^{-1}(a) \int_N f(ua)\, d\mu_N(u)\, d\mu_A(a),$$

where $\delta_B^{-1}(\mathrm{diag}(a_1, \ldots, a_n)) = \prod_{1 \leq i < j \leq n} \left|\frac{a_j}{a_i}\right|$, and we get the decomposition

$$\int_G f(g)\, d\mu_G(g) = \int_A \int_N \int_K \delta_B^{-1}(a) \cdot f(uak)\, d\mu_K(k)\, d\mu_N(u)\, d\mu_A(a).$$

From the uniqueness of the measure $\mu_{{}_N\backslash^G}$ (see Theorem 1.3), we conclude that for $f \in C^\infty\left({}_N\backslash^G\right)$

$$\int_{{}_N\backslash^G} f(g)\, d\mu_{{}_N\backslash^G}(g) = \int_A \int_K \delta_B^{-1}(a) \cdot f(ak)\, d\mu_K(k)\, d\mu_A(a).$$

3.1.3. *Local zeta integrals.*

Theorem 3.3 (Local zeta integrals of Tate)**.** *Let $\chi : F^* \to \mathbb{C}^*$ be a unitary character of F and let $\phi \in \mathcal{S}(F)$, $s \in \mathbb{C}$.*

(1) *The integral*

$$Z(s,\phi,\chi) = \int_{F^*} \phi(x)\,\chi(x)\,|x|^s\, d\mu_{F^*}(x)$$

converges absolutely for $\mathrm{Re}(s) > 0$*. It converges to an element of* $\mathbb{C}(q^s)$ *and therefore has a meromorphic continuation to the entire complex plane.*

(2) *Define* $L(s,\chi) = \begin{cases} \frac{1}{1-\chi(\varpi)q^{-s}} & \chi \text{ is unramified } (\chi\restriction_{\mathcal{O}^*}\equiv 1) \\ 1 & \chi \text{ is ramified} \end{cases}$*. Then*

$$\{Z(s,\phi,\chi) \mid \phi \in \mathcal{S}(F)\} = L(s,\chi)\cdot\mathbb{C}\left[q^{-s},q^s\right].$$

(See [GH11, Remark 2.3.3, Theorem 2.3.13, Theorem 2.4.13]*).*

Theorem 3.4 (Local zeta integrals of Godement and Jacquet)**.** *Let π be an irreducible smooth representation of $G = \mathrm{GL}_n(F)$, $\phi \in \mathcal{S}(M_n(F))$, $s \in \mathbb{C}$. Let $f : G \to \mathbb{C}$ be a matrix coefficient of π, i.e. $f(g) = f_{v,\tilde{v}}(g) = \langle \tilde{v}, \pi(g)v\rangle$ for $v \in V_\pi$, $\tilde{v} \in \widetilde{V_\pi}$.*

(1) *There exists some $r_\pi \in \mathbb{R}$ depending on π only such that the integral*

$$Z(s,\phi,f) = \int_G \phi(g)\, f(g)\, |\det g|^s\, d\mu_G(g)$$

converges absolutely for $\mathrm{Re}(s) > r_\pi$*. It converges to an element of* $\mathbb{C}(q^s)$ *and therefore has a meromorphic continuation to the entire complex plane.*

(2) *There exists a unique element $p(X) \in \mathbb{C}[X]$ with $p(0) = 1$ such that*

$$\left\{Z\left(s+\frac{n-1}{2},\phi,f_{v,\tilde{v}}\right) \mid \phi \in \mathcal{S}(F), v \in V_\pi, \tilde{v} \in \widetilde{V_\pi}\right\} = \frac{1}{p(q^{-s})}\cdot\mathbb{C}\left[q^{-s},q^s\right].$$

We denote $L(\pi,s) = \frac{1}{p(q^{-s})}$.

(See [GJ72, Page 30, Theorem 3.3]).

Theorem 3.5. *Let π be an irreducible smooth supercuspidal representation of $\mathrm{GL}_n(F)$, where $n > 1$. Then $L(\pi,s) = 1$.* [Jac79, Example 1.3.5]

3.1.4. *Estimates on Whittaker functions.* Let $a_1,\dots,a_{n-1} \in F^*$. We denote

$$m(a_1,a_2,\dots,a_{n-1}) = \mathrm{diag}(a_1a_2\cdot\dots\cdot a_{n-1}, a_2\cdot\dots\cdot a_{n-1},\dots,a_{n-2}a_{n-1},a_{n-1},1).$$

Proposition 3.6. *Let π be a generic irreducible representation of $\mathrm{GL}_n(F)$. Let $W \in \mathcal{W}(\pi,\psi)$. Define $f : (F^*)^{n-1} \to \mathbb{C}$ by*

$$f(a_1,\dots,a_{n-1}) = W(m(a_1,a_2,\dots,a_{n-1})).$$

Then f is locally constant. Furthermore, for every $1 \le i_0 \le n-1$ there exists $R_{i_0} > 0$, such that $f(a_1, \dots, a_{n-1}) = 0$ for $a_1, \dots, a_{n-1} \in F^*$ having $|a_{i_0}| > R_{i_0}$.

Proof. Since π is smooth, there exists an open subgroup $U \subseteq G$, such that for every $g \in G$ and $u \in U$, we have $W(gu) = W(g)$. Intersecting with the diagonal subgroup of G yields a subgroup of the form $A \cap U = \{\operatorname{diag}(b_1, b_2, \dots, b_n)\}$, where $b_1, \dots, b_n$ belong to open subgroups of F^*. From continuity of the map

$$(a_1, \dots, a_{n-1}) \mapsto m(a_1, \dots, a_{n-1})$$

we get that the set

$$U' = \left\{ \left(\frac{b_1}{b_2}, \frac{b_2}{b_3}, \dots, \frac{b_{n-2}}{b_{n-1}}, b_{n-1} \right) \mid \operatorname{diag}(b_1, b_2, \dots, b_{n-1}, 1) \in U \right\}$$

is open. Since W is invariant to right translations by elements of U, f is invariant to multiplication by elements of U'. Therefore f is locally constant.

Let $K_M = I_n + \varpi^M M_n(\mathcal{O})$ be a congruence subgroup of $\mathrm{GL}_n(\mathcal{O})$, such that W is invariant under right translations of K_M.

Let $1 \le i_0 \le n-1$. Consider the unipotent radical associated to the partition $(i_0, n-i_0)$:

$$N_{(i_0, n-i_0)} = \left\{ \left(\begin{array}{c|c} I_{i_0} & * \\ \hline 0_{(n-i_0)\times i_0} & I_{n-i_0} \end{array} \right) \right\}.$$

Then for every element $u \in K_M \cap N_{(i_0,n-i_0)}$ and $g \in G_{2m}$ we have $W(gu) = W(g)$. On the other hand, taking $g = \operatorname{diag}(t_1, \dots, t_n)$ yields $gug^{-1} \in N_{(i_0,n-i_0)}$ and therefore

$$W(gu) = W\left(\left(gug^{-1}\right) g\right) = \psi\left(gug^{-1}\right) W(g).$$

Since $u \in N_{(i_0,n-i_0)}$, the element gug^{-1} has zeros above its diagonal, except for the place (i_0, i_0+1), where it has the value $\frac{t_{i_0}}{t_{i_0+1}} u_{i_0,i_0+1}$. Therefore

$$W(gu) = \psi\left(\frac{t_{i_0}}{t_{i_0+1}} u_{i_0,i_0+1}\right) W(g),$$

and we get that $W(g) = \psi\left(\frac{t_{i_0}}{t_{i_0+1}} u_{i_0,i_0+1}\right) W(g)$, for every $u \in K_M \cap N_{(i_0,n-i_0)}$. Suppose $\psi \restriction_{\mathcal{P}^{N_0}} \equiv 1$ and $\psi \restriction_{\mathcal{P}^{N_0-1}} \not\equiv 1$ (i.e. $\mathcal{P}^{N_0} = \varpi^{N_0}\mathcal{O}$ is the conductor of ψ). If $\left|\frac{t_{i_0}}{t_{i_0+1}}\right| > q^{-N_0} \cdot q^M$, then we can choose an element $u \in K_M \cap N_{(i_0,n-i_0)}$, such that $\psi\left(\frac{t_{i_0}}{t_{i_0+1}} u_{i_0,i_0+1}\right) \neq 1$ (by choosing a suitable $|u_{i_0,i_0+1}| \le q^{-M}$ and placing zeros in other non-diagonal entries), and therefore from the equality $W(g) = \psi\left(\frac{t_{i_0}}{t_{i_0+1}} u_{i_0,i_0+1}\right) W(g)$, we have that $W(g) = 0$. Translating this to f, we get that $f(a_1, a_2, \dots, a_{n-1}) = 0$ for $|a_{i_0}| > R_{i_0}$, where $R_{i_0} = q^{-N_0} \cdot q^M$. □

Proposition 3.7. *Let π be a generic irreducible supercuspidal representation. Let $W \in \mathcal{W}(\pi, \psi)$ be a Whittaker function. Define f as above. Then $f \in \mathcal{S}\left((F^*)^{n-1}\right)$.*

Proof. It follows from the previous proposition that f is locally constant and vanishes whenever $|a_i|$ is large for some $1 \le i \le n-1$. We show that f vanishes whenever $|a_i|$ is small, for some $1 \le i \le n-1$. Combining with the previous result, this yields

$$\operatorname{supp} f \subseteq \{(a_1, \dots, a_{n-1}) \mid \forall 1 \le i \le n,\ r \le |a_i| \le R\},$$

where $r, R > 0$. The right hand side set is a compact subset of $(F^*)^{n-1}$ and therefore $\mathrm{supp} f$ is compact as a closed subset of $(F^*)^{n-1}$ contained in a compact set.

Since π is supercuspidal,

$$\mathcal{W}(\pi, \psi) = \mathrm{span}_{\mathbb{C}} \left\{ \rho(u) W' - W' \mid u \in N_\alpha, W' \in \mathcal{W}(\pi, \psi) \right\},$$

where $\alpha \neq (n)$ is a partition of n and N_α is the unipotent radical of $\mathrm{GL}_n(F)$ corresponding to α (This is true for any partition $\alpha \neq (n)$).

Let $1 \leq i_0 \leq n-1$. Taking $\alpha = (i_0, n - i_0)$ we get that

$$W = \sum_{i=1}^{l} \left(\rho\left(u^{(i)}\right) W_i - W_i \right),$$

where $l \geq 0$, $(W_i)_{i=1}^{l} \subseteq \mathcal{W}(\pi, \psi)$ and $\left(u^{(i)}\right)_{i=1}^{l} \subseteq N_{(i_0, n-i_0)}$. For every $g \in G$ we have

$$W(g) = \sum_{i=1}^{l} \left(W_i\left(g u^{(i)}\right) - W_i(g) \right).$$

Taking $g = \mathrm{diag}(t_1, \dots, t_n)$ as before yields

$$\begin{aligned} W(g) &= \sum_{i=1}^{l} \left(W_i\left(g u^{(i)} g^{-1} g\right) - W_i(g) \right) \\ &= \sum_{i=1}^{l} \left(\psi\left(\frac{t_{i_0}}{t_{i_0+1}} u^{(i)}_{i_0, i_0+1} \right) - 1 \right) W_i(g). \end{aligned}$$

Suppose that $\psi \restriction_{\mathcal{P}^N} \equiv 1$ and $\psi \restriction_{\mathcal{P}^{N-1}} \not\equiv 1$ (i.e. $\mathcal{P}^N$ is the conductor of ψ). Therefore if $\left| \frac{t_{i_0}}{t_{i_0+1}} u^{(i)}_{i_0, i_0+1} \right| \leq q^{-N}$ for every $1 \leq i \leq l$, i.e.

$$\left| \frac{t_{i_0}}{t_{i_0+1}} \right| \cdot \max_{i=1}^{l} \left| u^{(i)}_{i_0, i_0+1} \right| \leq q^{-N}.$$

Then we have $\psi\left(\frac{t_{i_0}}{t_{i_0+1}} u^{(i)}_{i_0, i_0+1} \right) = 1$, for every $1 \leq i \leq l$, and therefore $W(g) = 0$. Translating this to f, we get that $f(a_1, \dots, a_{n-1}) = 0$ for $a_1, \dots, a_{n-1} \in F^*$ having $|a_{i_0}| \leq r_{i_0}$, where $r_{i_0} = \frac{q^{-N}}{\max\left\{1, \max_{i=1}^{l} \left| u^{(i)}_{i_0, i_0+1} \right| \right\}}$. □

Proposition 3.8. *Let G be an l-group and π be a smooth representation of G. Suppose that $\alpha : X \to G$ is a continuous map where X is a compact topological space. Let $v \in V_\pi$, then there exist a finite number of independent vectors $(v_i)_{i=1}^{N}$ and smooth functions $(\alpha_i)_{i=1}^{N}$ with $\alpha_i : X \to \mathbb{C}$ such that*

$$\pi(\alpha(x)) v = \sum_{i=1}^{N} \alpha_i(x) v_i.$$

Proof. Since α is continuous, $\alpha(X) \subseteq G$ is compact. Since π is smooth, $\mathrm{stab}_G v$ is open, and therefore the cover $\alpha(X) \subseteq \bigcup_{x \in X} \alpha(x) \cdot \mathrm{stab}_G v$ has a finite sub-cover

$$\alpha(X) \subseteq \bigcup_{i=1}^{M} \alpha(x_i) \cdot \mathrm{stab}_G v.$$

Therefore

$$\operatorname{span}_{\mathbb{C}}\left\{\pi\left(\alpha\left(x\right)\right)v \mid x \in X\right\} \subseteq \operatorname{span}_{\mathbb{C}}\left\{\pi\left(\alpha\left(x_i\right)\right)v \mid 1 \leq i \leq M\right\}$$

is finite dimensional. Choose a basis $(v_i)_{i=1}^N$ for $\operatorname{span}_{\mathbb{C}}\left\{\pi\left(\alpha\left(x\right)\right)v \mid x \in X\right\}$. Therefore for every $x \in X$ there exist $(\alpha_i(x))_{i=1}^N \subseteq \mathbb{C}$ such that

$$\pi\left(\alpha\left(x\right)\right)v = \sum_{i=1}^{N} \alpha_i\left(x\right)v_i.$$

We show that α_i are smooth functions.

Let $x_0 \in X$. Since $\operatorname{stab}_G v$ is open, so is $\alpha(x_0) \cdot \operatorname{stab}_G v$. Therefore, from continuity, the inverse image $\alpha^{-1}\left(\alpha\left(x_0\right) \cdot \operatorname{stab}_G v\right)$ is open. Denote this set as U_{x_0}. For every $x \in U_{x_0}$, we have $\alpha(x) \in \alpha(x_0) \cdot \operatorname{stab}_G v$, and therefore $\pi(\alpha(x))v = \pi(\alpha(x_0))v$, which implies

$$\sum_{i=1}^{N} \alpha_i\left(x_0\right)v_i = \sum_{i=1}^{N} \alpha_i\left(x\right)v_i.$$

Since $(v_i)_{i=1}^N$ are independent, $\alpha_i(x_0) = \alpha_i(x)$, for every $1 \leq i \leq N$. We have shown that for every $1 \leq i \leq N$, $\alpha_i(x_0) = \alpha_i(x)$, for every $x \in U_{x_0}$, and therefore $(\alpha_i)_{i=1}^N$ are smooth. □

Using Propositions 3.7 and 3.8 (with $G = X = \mathrm{GL}_n(\mathcal{O})$, $\alpha = \mathrm{id}$) we obtain the following:

Corollary 3.9. *Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$ and let $W \in \mathcal{W}(\pi, \psi)$. Then for $a = m(a_1, \ldots, a_{n-1})$ and $k \in \mathrm{GL}_n(\mathcal{O})$ the function $f(a_1, \ldots, a_{n-1}, k) = W(ak)$ is an element of $\mathcal{S}\left(\left(F^*\right)^{n-1} \times \mathrm{GL}_n(\mathcal{O})\right)$.*

Proof. Using Proposition 3.8 we write $W(ak) = \sum_{i=1}^N \alpha_i(k) W_i(a)$, where $\alpha_i : \mathrm{GL}_n(\mathcal{O}) \to \mathbb{C}$ are smooth. Since $\mathrm{GL}_n(\mathcal{O})$ is compact, $(\alpha_i)_{i=1}^N$ are Schwartz functions. We then use Proposition 3.7 to obtain that $f_i \in \mathcal{S}\left(\left(F^*\right)^{n-1}\right)$, where $f_i(a_1, \ldots, a_{n-1}) = W_i(m(a_1, \ldots, a_{n-1}))$, and the corollary follows. □

3.1.5. *Finite functions.* Before stating the asymptotic expansion of Whittaker functions in the general case (where π isn't necessarily supercuspidal), we shortly review the topic of finite functions of $(F^*)^n$. We will mainly need Proposition 3.11.

Definition 3.10. Let G be an Abelian l-group. A finite function $f : G \to \mathbb{C}$ is a smooth function such that the translations of f span a finite dimensional space.

Proposition 3.11. *$f : (F^*)^n \to \mathbb{C}$ is a finite function if and only if*

$$f \in \operatorname{span}_{\mathbb{C}}\left\{\prod_{i=1}^{n} \chi_i\left(a_i\right)\log^{m_i}\left|a_i\right| \mid 0 \leq m_i \in \mathbb{Z},\ \chi_i : F^* \to \mathbb{C}^* \textit{ is a character of } F^*\right\}.$$

(See [JL70, Section 8]*).*

Recall that every character $\chi : F^* \to \mathbb{C}^*$ can be written uniquely in the form $\chi(a) = |a|^{r_\chi} \cdot \omega_\chi(a)$ where $r_\chi \in \mathbb{R}$ and $\omega_\chi : F^* \to \mathbb{C}^*$ is a unitary character. We denote $\Re(\chi) = r_\chi$.

3.1.6. *Asymptotic expansion of Whittaker functions in the general case.*

Proposition 3.12. *Let π be a generic irreducible representation of $\mathrm{GL}_n(F)$. Then there exist finite functions $(\xi_i)_{i=1}^t$ on $(F^*)^{n-1}$, such that for any $W \in \mathcal{W}(\pi,\psi)$ there are t functions $(\phi_i)_{i=1}^t \subseteq \mathcal{S}(F^{n-1})$, such that*

$$W(m(a_1,\dots,a_{n-1})) = \sum_{i=1}^t \xi_i(a_1,\dots,a_{n-1}) \cdot \phi_i(a_1,\dots,a_{n-1}),$$

where $a = m(a_1,\dots,a_{n-1})$ (See [JPSS79, Proposition 2.2]*).*

Consider the Haar modular function of the Borel subgroup $B_{n-1} \subseteq \mathrm{GL}_{n-1}(F)$, $\delta_{B_{n-1}} : A_{n-1} \to \mathbb{C}$, $\delta_B^{-1}(\mathrm{diag}(a_1,\dots,a_{n-1})) = \prod_{1\le i<j\le n-1} \left|\frac{a_j}{a_i}\right|$. The function $\delta_{B_{n-1}}^{\frac{1}{2}}$ is a non-vanishing finite function (it is a positive character) and therefore by modifying the set $(\xi_i)_{1\le i\le t}$ in Proposition 3.12, it is clear that one can write

$$W(a) = \delta_{B_{n-1}}^{\frac{1}{2}}(a) \sum_{i=1}^t \xi_i(a_1,\dots,a_{n-1}) \cdot \phi_i(a_1,\dots,a_{n-1}),$$

where $a = m(a_1,\dots,a_{n-1})$ and $\phi_i \in \mathcal{S}(F^{n-1})$.

Furthermore, from Proposition 3.11, there exist finite sets $(C_j)_{j=1}^{n-1}$ of characters $\chi : F^* \to \mathbb{C}^*$ and non-negative integers $(r_j)_{j=1}^{n-1}$, such that

$$(\xi_i)_{i=1}^t \subseteq \mathrm{span}_{\mathbb{C}} \left\{ \chi(a_1,\dots,a_{n-1}) = \prod_{j=1}^{n-1} \chi_j(a_j) \log^{m_j}|a_j| \mid \chi_j \in C_j,\ m_j \in \mathbb{Z} \mid 0 \le m_j \le r_j \right\}.$$

Denote for such sets and integers

$$X = X_{(r_j,C_j)_{j=1}^{n-1}} = \left\{ \chi(a_1,\dots,a_{n-1}) = \prod_{j=1}^{n-1} \chi_j(a_j) \log^{m_j}|a_j| \mid \chi_j \in C_j,\ m_j \in \mathbb{Z} \mid 0 \le m_j \le r_j \right\}.$$

We may assume that $\{\xi_i \mid 1 \le i \le t\} = X$, as X spans the original set.

Finally, using Proposition 3.8 (as in Corollary 3.9), we obtain the following:

Proposition 3.13. *Let π be a generic irreducible representation of $\mathrm{GL}_n(F)$. Then for each $1 \le j \le n-1$, there exist an integer r_j and a finite set C_j of characters $\chi : F^* \to \mathbb{C}^*$, such that for $X = X_{(r_j,C_j)_{j=1}^{n-1}}$ and for any $W \in \mathcal{W}(\pi,\psi)$, there are functions $(\phi_\xi)_{\xi\in X} \subseteq \mathcal{S}(F^{n-1} \times \mathrm{GL}_n(\mathcal{O}))$, such that*

$$W(ak) = \delta_{B_{n-1}}^{\frac{1}{2}}(a) \sum_{\xi\in X} \xi(a_1,\dots,a_{n-1}) \cdot \phi_\xi(a_1,\dots,a_{n-1},k),$$

for every $a = m(a_1,\dots,a_{n-1})$, and $k \in \mathrm{GL}_n(\mathcal{O})$.

Remark 3.14. One can show that if π is a generic irreducible unitary representation of $\mathrm{GL}_n(F)$, then the sets C_j can be chosen, such that for every $\chi \in C_j$, $\Re(\chi) > 0$. [JS90, Section 4, Proposition 3]

3.2. **Convergence.** Before proving that $J_{\pi,\psi}$ converges absolutely for s in a right half plane, we prove some statements used throughout the proof.

3.2.1. *Theorems regarding the diagonal part of an Iwasawa decomposition of u_Z.* We will need the following theorem regarding the diagonal part of the Iwasawa decomposition of some matrix.

We follow [JS90, Section 5, Propositions 4, 5].

Theorem 3.15. *Let $Z \in M_m(F)$ be a lower triangular nilpotent matrix and $u_Z = w_{m,m} \left(\begin{smallmatrix} I_m & Z \\ & I_m \end{smallmatrix}\right) w_{m,m}^{-1}$. Suppose $u_Z = n_Z t_Z k_Z$ is an Iwasawa decomposition of u_Z (i.e. $n_Z \in N_{2m}$, $t_Z \in A_{2m}$, $k_Z \in K_{2m}$). Write $t_Z = \operatorname{diag}(t_1, \dots, t_{2m})$. Then $|t_i| \geq 1$ for odd i and $|t_i| \leq 1$ for even i. Furthermore $|t_1| = |t_{2m}| = 1$.*

Before proving this theorem, we discuss some properties of the maximum norm of the exterior power of the space spanned by row elements $(e_i)_{i=1}^n$.

Let V be a finite dimensional vector space over F. Let $\{v_1, \dots, v_d\}$ be a basis for V. For every $1 \leq r \leq d$, we define a norm on $\Lambda^r(V)$, the r-th exterior power of V, by

$$\left\| \sum_{1 \leq i_1 < \dots < i_r \leq d} a_{i_1 i_2 \dots i_r} v_{i_1} \wedge \dots \wedge v_{i_r} \right\| = \max_{1 \leq i_1 < \dots < i_r \leq d} |a_{i_1 i_2 \dots i_r}|.$$

Remark 3.16. Note that for $v \in V$, $v = \sum_{i=1}^d b_i v_i$ we have $\|v\| = \max_{1 \leq i \leq d} |b_i|$ (here $r = 1$).

Claim 3.17. This norm has the property that for every $1 \leq r \leq d-1$, $\alpha \in V^r(V)$ and $v \in V$, the following inequality holds:

$$\|v \wedge \alpha\| \leq \|v\| \, \|\alpha\|.$$

Proof. Write $v = \sum_{j=1}^d b_j v_j$ and $\alpha = \sum_{1 \leq i_1 < \dots < i_r \leq d} a_{i_1 i_2 \dots i_r} v_{i_1} \wedge \dots \wedge v_{i_r}$, where $a_{i_1 i_2 \dots i_r}, b_j \in F$. Then

$$v \wedge \alpha = \sum_{j=1}^d \sum_{1 \leq i_1 < \dots < i_r \leq d} b_j a_{i_1 i_2 \dots i_r} v_j \wedge v_{i_1} \wedge \dots \wedge v_{i_r}.$$

We get that the coefficients of $v \wedge \alpha$ are sums of the form $\sum (-1)^s b_j a_{i_1 i_2 \dots i_r}$. These have absolute value

$$\left| \sum (-1)^s b_j a_{i_1 i_2 \dots i_r} \right| \leq \max_{j \notin \{i_1, \dots, i_r\}} |b_j| \, |a_{i_1 \dots i_r}| \leq \max_{1 \leq j \leq d} |b_j| \max_{1 \leq i_1 < \dots < i_r \leq d} |a_{i_1 \dots i_r}| = \|v\| \cdot \|\alpha\|,$$

and therefore the norm of $v \wedge \alpha$, which is the maximal absolute value of the coefficients of $v \wedge \alpha$, is not greater than $\|v\| \cdot \|\alpha\|$. □

We now take V to be the space spanned by the row vectors $(e_i)_{i=1}^n \subseteq F^{1 \times n}$.

Proposition 3.18. *For a matrix $k \in K_n = \mathrm{GL}_n(\mathcal{O})$ and $1 \leq r \leq n$, we have*

$$\|(e_r k) \wedge (e_{r+1} k) \wedge \dots \wedge (e_n k)\| = 1.$$

Proof. All matrix elements of k are in $\mathcal{O}$ and therefore have absolute value ≤ 1. Hence $\|e_i k\| \leq 1$. By using the inequality $\|v \wedge \alpha\| \leq \|v\| \, \|\alpha\|$ repeatedly, one gets

$$\begin{aligned} \|(e_r k) \wedge (e_{r+1} k) \wedge \dots \wedge (e_n k)\| &\leq \underbrace{\|e_r k\|}_{\leq 1} \|(e_{r+1} k) \wedge \dots \wedge (e_n k)\| \\ &\leq \|(e_{r+1} k) \wedge \dots \wedge (e_n k)\| \leq \dots \leq \|e_n k\| \leq 1. \end{aligned}$$

On the other hand

$$(e_1 k) \wedge \cdots \wedge (e_n k) = \det k \cdot (e_1 \wedge \cdots \wedge e_n),$$

and therefore

$$\|(e_1 k) \wedge \cdots \wedge (e_n k)\| = |\det k| \cdot \|e_1 \wedge \cdots \wedge e_n\| = 1,$$

which implies

$$1 = \|(e_1 k) \wedge \cdots \wedge (e_n k)\| \le \|(e_2 k) \wedge \cdots \wedge (e_n k)\| \le \cdots \le \|(e_r k) \wedge \cdots \wedge (e_n k)\|,$$

and we get the desired equality $\|(e_r k) \wedge \cdots \wedge (e_n k)\| = 1$. □

Corollary 3.19. *For every $k \in K_n$ and every $1 \le i \le n$, we have $\|e_i k\| = 1$.*

Proof. We have seen already that $1 = \|(e_1 k) \wedge \cdots \wedge (e_n k)\|$. On the other hand, as in the previous proof

$$1 = \|(e_1 k) \wedge \cdots \wedge (e_n k)\| \le \|e_1 k\| \cdot \cdots \cdot \|e_n k\| \le 1,$$

hence

$$\|e_1 k\| \cdot \cdots \cdot \|e_n k\| = 1.$$

Combining this with the fact that $\|e_i k\| \le 1$, for all $1 \le i \le n$ (since the entries of k are in $\mathcal{O}$), implies $\|e_i k\| = 1$, for all $1 \le i \le n$. □

Proposition 3.20. *Let $u_Z = n_Z t_Z k_Z$ where $n_Z \in N_n$ $t_Z = \operatorname{diag}(t_1, \ldots, t_n) \in A_n$, $k_Z \in K_n$ and let $1 \le r \le n$. Then $\|(e_r u_Z) \wedge \cdots \wedge (e_n u_Z)\| = |t_r t_{r+1} \cdot \cdots \cdot t_n|$.*

Proof. Write

$$(e_r u_Z) \wedge \cdots \wedge (e_n u_Z) = (e_r n_Z t_Z k_Z) \wedge \cdots \wedge (e_n n_Z t_Z k_Z).$$

Denote $T_{n_Z}, T_{t_Z}, T_{k_Z} : V \to V$ the maps $T_{n_Z}(v) = v n_Z$, $T_{t_Z}(v) = v t_Z$, $T_{k_Z}(v) = v k_Z$. Then the above wedge product equals

$$\begin{aligned}(e_r n_Z t_Z k_Z) \wedge \cdots \wedge (e_n n_Z t_Z k_Z) &= (T_{k_Z} T_{t_Z} T_{n_Z} e_r) \wedge \cdots \wedge (T_{k_Z} T_{t_Z} T_{n_Z} e_n) \\ &= \Lambda^{n-r+1} T_{k_Z} \Lambda^{n-r+1} T_{t_Z} \left((T_{n_Z} e_r) \wedge \cdots \wedge (T_{n_Z} e_n) \right).\end{aligned}$$

We notice that the subspace V_r spanned by $\{e_r, \ldots, e_n\}$ is invariant under T_{n_Z}. The matrix of $T_{n_Z} \restriction_{V_r}$, with respect to the basis $\{e_r, \ldots, e_n\}$, is the transpose of the sub-matrix of n_Z consisting of its last $n - r + 1$ rows and columns. Therefore the restriction of T_{n_Z} to V_r has determinant 1 and we have

$$(T_{n_Z} e_r) \wedge \cdots \wedge (T_{n_Z} e_n) = \det T_{n_Z} \restriction_{V_r} \cdot (e_r \wedge \cdots \wedge e_n) = e_r \wedge \cdots \wedge e_n.$$

Thus

$$\Lambda^{n-r+1} T_{k_Z} \Lambda^{n-r+1} T_{t_Z} \left((T_{n_Z} e_r) \wedge \cdots \wedge (T_{n_Z} e_n) \right) = \Lambda^{n-r+1} T_{k_Z} \Lambda^{n-r+1} T_{t_Z} (e_r \wedge \cdots \wedge e_n).$$

Since $T_{t_Z} e_i = e_i t_Z = t_i e_i$ and $\Lambda^{n-r+1} T_{k_Z} (e_r \wedge \cdots \wedge e_n) = (e_r k_Z) \wedge \cdots \wedge (e_n k_Z)$, we get

$$\Lambda^{n-r+1} T_{k_Z} \Lambda^{n-r+1} T_{t_Z} (e_r \wedge \cdots \wedge e_n) = t_r t_{r+1} \cdot \cdots \cdot t_n \cdot \left((e_r k_Z) \wedge \cdots \wedge (e_n k_Z) \right).$$

Taking $\|\cdot\|$, we get

$$\|(e_r u_Z) \wedge \cdots \wedge (e_n u_Z)\| = |t_r t_{r+1} \cdot \cdots \cdot t_n| \, \|(e_r k_Z) \wedge \cdots \wedge (e_n k_Z)\| = |t_r t_{r+1} \cdot \cdots \cdot t_n|,$$

where the last step uses the previous proposition. □

We now move to the proof of Theorem 3.15.

Proof. We first write the form of the matrix $u_Z = w_{m,m} \left(\begin{smallmatrix} I_m & Z \\ & I_m \end{smallmatrix}\right) w_{m,m}^{-1}$ where $Z \in M_m(F)$ is a lower triangular nilpotent matrix. We recall that for an arbitrary $(a_{ij})_{i,j}$ we have $w_{m,m}(a_{ij})\, w_{m,m}^{-1} = \left(a_{\sigma^{-1}(i)\sigma^{-1}(j)}\right)_{i,j}$ where σ is the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & m & m+1 & m+2 & m+3 & \dots & 2m \\ 1 & 3 & 5 & \dots & 2m-1 & 2 & 4 & 6 & \dots & 2m \end{pmatrix}$$

The diagonal of $\left(\begin{smallmatrix} I_m & Z \\ & I_m \end{smallmatrix}\right)$ consists of the diagonal of the identity matrix I_{2m}. It is preserved under conjugation. We compute which non-diagonal entries of u_Z can be non-zero. These are elements having index (i,j), with $(\sigma^{-1}(i), \sigma^{-1}(j)) = (i', m+j')$, with $1 \le i' \le m$ and $1 \le j' \le m$ and $i' > j'$, i.e. $i = \sigma(i') = 2i'-1$, $j = 2j'$. We notice that $i > j$, since $i' \ge j'+1$, i.e. u_Z is a lower triangular unipotent matrix. We also get that u_Z has the row vector e_i as its i-th row for even i. Similarly, u_Z has the column vector e_i^t as its i-th column for odd i. We illustrate the shape of u_Z by writing it for $m = 4$: (The matrix has zeros above its diagonal)

$$u_Z = \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & * & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & * & 0 & * & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & \\ 0 & * & 0 & * & 0 & * & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have shown that $u_Z e_i = e_i$ for even i. By the previous claim, we have that for even i

$$\|(e_i u_Z) \wedge \cdots \wedge (e_{2m} u_Z)\| = |t_i t_{i+1} \cdot \dots \cdot t_{2m}|$$
$$\|e_i \wedge (e_{i+1} u_Z) \wedge \cdots \wedge (e_{2m} u_Z)\| = |t_i t_{i+1} \cdot \dots \cdot t_{2m}|.$$

From the inequality $\|v \wedge \alpha\| \le \|v\| \|\alpha\|$, we get

$$\|e_i \wedge (e_{i+1} u_Z) \wedge \cdots \wedge (e_{2m} u_Z)\| \le \|(e_{i+1} u_Z) \wedge \cdots \wedge (e_{2m} u_Z)\| = |t_{i+1} \cdot \dots \cdot t_{2m}|,$$

and hence $|t_i| \le 1$, and the theorem is proved for even i.

In order to prove the theorem for odd i, we write $u_Z = n_Z t_Z k_Z$ and therefore

$$k_Z = t_Z^{-1} n_Z^{-1} u_Z.$$

For odd i, we have seen that the ith column of u_Z is the column e_i^t, which implies that the ith column of $n_Z^{-1} u_Z$ is the same as the ith column of n_Z^{-1}. This implies that for odd i, $k_Z = t_Z^{-1} n_Z^{-1} u_Z$ has the value t_i^{-1} in the i-th place on the diagonal. Since $k_Z \in K_{2m} = \mathrm{GL}_{2m}(\mathcal{O})$, we get for odd i, $\left|t_i^{-1}\right| \le 1$, i.e. $|t_i| \ge 1$, as required.

As for t_1 and t_{2m}: since $e_{2m} = e_{2m} u_Z = e_{2m} n_Z t_Z k_Z = t_{2m} e_{2m} k_Z$ we have

$$1 = \|e_{2m}\| = \|t_{2m} e_{2m} k_Z\| = |t_{2m}| \|e_{2m} k_Z\| = |t_{2m}| \|e_{2m}\| = |t_{2m}|.$$

Regarding t_1, write $k_Z e_1^t = t_Z^{-1} n_Z^{-1} u_Z e_1^t = t_Z^{-1} e_1^t = t_1^{-1} e_1^t$, and therefore $e_1 k_Z^t = t_1^{-1} e_1$. And since $k_Z^t \in K_{2m}$, this implies

$$1 = \|e_1\| = \left\|e_1 k_Z^t\right\| = \left\|t_1^{-1} e_1\right\| = \left|t_1^{-1}\right| \|e_1\| = |t_1|^{-1},$$

as required. $\square$

Proposition 3.21. *Let $Z \in M_m(F)$ be a lower triangular nilpotent matrix and $u_Z = w_{m,m}\left(\begin{smallmatrix} I_m & Z \\ & I_m \end{smallmatrix}\right) w_{m,m}^{-1}$. Suppose $u_Z = n_Z t_Z k_Z$ is an Iwasawa decomposition of u_Z (i.e. $n_Z \in N_{2m}$, $t_Z \in A_{2m}$, $k_Z \in K_{2m}$). Write $t_Z = \operatorname{diag}(t_1, \ldots, t_{2m})$. Denote by $\|Z\|$ the maximum norm of Z. Then*

$$\max(1, \|Z\|)^{\frac{1}{2m}} \leq \prod_{\substack{1 \leq i \leq 2m \\ i \text{ is odd}}} |t_i| .$$

Proof. Denote for $1 \leq k \leq 2m$, $s_k = \|(e_k u_Z) \wedge \cdots \wedge (e_{2m} u_Z)\|$. By Proposition 3.20, $s_k = |t_k \cdot \cdots \cdot t_{2m}|$. The element $(e_k u_Z) \wedge \cdots \wedge (e_{2m} u_Z)$ is equal to the sum

$$(e_k u_Z) \wedge \cdots \wedge (e_{2m} u_Z) = \sum_{i_1 < \cdots < i_{2m-k+1}} a_{i_1 \ldots i_{2m-k+1}} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{2m-k+1}} .$$

By writing $e_i u_Z$ as a linear combination of $\{e_i, \ldots, e_{2m}\}$ using the coefficients of u_Z, we see that the coefficient $a_{i_1 \ldots i_{2m-k+1}}$ equals to the minor of u_Z consisting of the last $2m-k+1$ rows and the columns $i_1, \ldots, i_{2m-k+1}$ columns. Because of the special shape of u_Z, we see that every non zero element of u_Z is such a minor with k odd: we take for an element at the kth row its column, and the last $n-k$ columns of the matrix - this gives a lower triangular matrix with a diagonal consisting only of 1 and our element, and therefore its determinant value is equal to our element.

Therefore, we get that for all k, $\|(e_k u_Z) \wedge \cdots \wedge (e_{2m} u_Z)\| \geq \|e_k u_Z\| \geq 1$ and

$$\prod_{1 \leq k \leq 2m} \|(e_k u_Z) \wedge \cdots \wedge (e_{2m} u_Z)\| \geq \max_{1 \leq k \leq 2m} |u_Z e_k| = \|u_Z\| .$$

Since u_Z consists of the same non-zero elements as Z, except for 1 on the diagonal, we have $\|u_Z\| = \max\{1, \|Z\|\}$, and we get

$$\prod_{1 \leq k \leq 2m} s_k \geq \max\{1, \|Z\|\} .$$

From the previous theorem, we have

$$s_k = |t_k \cdot \cdots \cdot t_{2m}| \leq \prod_{\substack{1 \leq j \leq 2m \\ j \text{ is odd}}} |t_j| .$$

Therefore, we get

$$\max\{1, \|Z\|\} \leq \left(\prod_{\substack{1 \leq j \leq 2m \\ j \text{ is odd}}} |t_j| \right)^{2m} ,$$

as required. $\square$

Proposition 3.22. *We can choose smooth functions $Z \mapsto n_Z$, $Z \mapsto t_Z$ and a continuous function $Z \mapsto k_Z$ from $M_m(F)$ to N_{2m}, A_{2m}, K_{2m} respectively, such that $n_Z t_Z k_Z = u_Z$ is an Iwasawa decomposition of u_Z, for every $Z \in M_m(F)$. Furthermore, one can choose these, such that $t_Z \in A_{2m-1}$ (i.e. $t_Z = \operatorname{diag}(t_1, t_2, \ldots, t_{2m-1}, 1)$).*

Proof. The cosets of ${}^{M_m(F)}/_{M_m(\mathcal{O})}$ form a cover of $M_m(F)$ of pairwise disjoint compact-open subsets. We choose a representative for each coset such that

$$M_m(F) = \dot{\bigcup_{i\in I}} (Z_i + M_m(\mathcal{O})).$$

Let $u_{Z_i} = n_i a_i k_i$ (where $n_i \in N_{2m}$, $a_i \in A_{2m}$, $k_i \in K_{2m}$). Then for $N \in M_m(\mathcal{O})$ we have

$$\begin{aligned} u_{Z_i+N} &= w_{m,m} \begin{pmatrix} I_m & Z_i \\ & I_m \end{pmatrix} \begin{pmatrix} I_m & N \\ & I_m \end{pmatrix} w_{m,m}^{-1} \\ &= u_{Z_i} \cdot u_N. \end{aligned}$$

Since $N \in M_m(\mathcal{O})$, we have that $u_N = w_{m,m} \left(\begin{smallmatrix} I_m & N \\ & I_m \end{smallmatrix}\right) w_{m,m}^{-1} \in K_{2m}$, and therefore $u_{Z_i+N} = n_i a_i (k_i u_N)$ is an Iwasawa decomposition.

We define for every $N \in M_m(\mathcal{O})$ and $i \in I$, $n_{Z_i+N} = n_i$, $t_{Z_i+N} = a_i$, $k_{Z_i+N} = k_i u_N$. Since $Z_i + M_m(\mathcal{O})$ is compact open, it is clear that we have constructed functions as required.

Regarding the last part - write $t_Z = \operatorname{diag}(t_1, \dots, t_{2m})$. By Theorem 3.15, $|t_{2m}| = 1$ and therefore by replacing k_Z with $\operatorname{diag}(1, 1, \dots, 1, t_{2m}) \cdot k_Z$ and t_Z with $t_Z \cdot \operatorname{diag}(1, 1, \dots, 1, t_{2m}^{-1})$, we get an Iwasawa decomposition with $t_Z \in A_{2m-1}$. It is clear that t_Z is still smooth after this modification. ☐

3.2.2. *Convergence proof.* We now prove a theorem regarding the convergence of the integral. We follow [JS90, Section 7, Proposition 1].

Theorem 3.23. *Let π be an irreducible generic representation of $\mathrm{GL}_{2m}(F)$. There exists a real number $r_{\pi,\wedge^2} \in \mathbb{R}$ such that the integral $J_{\pi,\psi}(z, W, \phi)$ converges absolutely for every $z \in \mathbb{C}$ with $\operatorname{Re}(z) > r_{\pi,\wedge^2}$, $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in \mathcal{S}(F^m)$.*

Proof. We can assume that π has a unitary central character: Suppose that the theorem has been proved for representations with a unitary character. We can write for $a \in F^*$, $\omega_\pi(a) = \chi(a) \cdot |a|^r$ where χ is unitary and $r = \Re(\omega_\pi) \in \mathbb{R}$. Then $\tau = \pi \cdot \det^{-\frac{r}{2m}}$ has χ as its central character and therefore τ has a unitary central character. Note that $J_{\tau,\psi}\left(z + \frac{r}{m}, W, \phi\right) = J_{\pi,\psi}(z, W, \phi)$ and therefore $J_{\pi,\psi}(z, W, \phi)$ converges for every z with $\operatorname{Re}(z) > r_{\tau,\wedge^2} - \frac{r}{m}$.

We suppose that π has a unitary central character. Denote $s = \operatorname{Re}(z)$. Using the Iwasawa decomposition $G_m = NAK$ where $N = N_m$ the unipotent matrix subgroup, $A = A_m$ the diagonal matrix subgroup and $K = K_m = \mathrm{GL}_m(\mathcal{O})$, we write (see also Subsection 3.1.2)

$$\begin{aligned} &\int_{N\backslash G} \left(\int_{\mathcal{B}\backslash M} \left| W \left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \right) \right| |\psi(-\mathrm{tr}(X))| \, dX \right) |\phi(\varepsilon g)| \, |\det g|^s \, dg \\ &= \int_A da \int_K dk \left(\delta_B^{-1}(a) \int_{\mathcal{B}\backslash M} \left| W \left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} ak & \\ & ak \end{pmatrix} \right) \right| dX \right) |\phi(\varepsilon ak)| \, |\det(ak)|^s, \end{aligned}$$

where $B = B_m = N_m A_m = A_m N_m$ is the upper triangular matrix subgroup of G_m.

Conjugating by $\begin{pmatrix} a & \\ & a \end{pmatrix}$ and identifying ${}_{\mathcal{B}}\backslash^{M}$ with lower triangular nilpotent subgroup of M, which we denote $\mathcal{N}^-$, the integral gets the form

$$\int_A da \int_K dk \int_{\mathcal{N}^-} dX \left(\delta_B^{-1}(a) \left| W \left(w_{m,m} \begin{pmatrix} a & \\ & a \end{pmatrix} \begin{pmatrix} I_m & a^{-1} X a \\ & I_m \end{pmatrix} \begin{pmatrix} k & \\ & k \end{pmatrix} \right) \right| \right) |\phi(\varepsilon ak)| \, |\det(a)|^s.$$

We write $a = \operatorname{diag}(a_1, \dots, a_m) = a_m I_m \cdot \operatorname{diag}\left(\frac{a_1}{a_m}, \frac{a_2}{a_m}, \dots, \frac{a_{m-1}}{a_m}, 1\right)$ and denote

$$a' = \operatorname{diag}\left(\frac{a_1}{a_m}, \frac{a_2}{a_m}, \dots, \frac{a_{m-1}}{a_m}, 1\right).$$

Then

$$W\left(w_{m,m}\begin{pmatrix} a & \\ & a\end{pmatrix}\begin{pmatrix} I_m & a^{-1}Xa \\ & I_m\end{pmatrix}\begin{pmatrix} k & \\ & k\end{pmatrix}\right) = \omega_\pi(a_m)\, W\left(w_{m,m}\cdot\begin{pmatrix} a' & \\ & a'\end{pmatrix}\cdot\begin{pmatrix} I_m & a^{-1}Xa \\ & I_m\end{pmatrix}\begin{pmatrix} k & \\ & k\end{pmatrix}\right).$$

Since ω_π is unitary, $|\omega_\pi(a_m)| = 1$. Using the following measure decomposition of A: $d\mu_{A_m}(a'a_m) = d\mu_{A_{m-1}}(a')\, d\mu_{F^*}(a_m)$ (where we think of $A_{m-1} \subseteq A_m$ by the embedding $\operatorname{diag}(a_1, \dots, a_{m-1}) \mapsto \operatorname{diag}(a_1, , \dots, a_{m-1}, 1)$), we get

$$\int_{A_{m-1}} da' \int_{F^*} da_m \int_K dk \int_{\mathcal{N}^-} dX \left(\delta_B^{-1}(a')\left|W\left(w_{m,m}\begin{pmatrix} a' & \\ & a'\end{pmatrix}\begin{pmatrix} I_m & a'^{-1}Xa' \\ & I_m\end{pmatrix}\begin{pmatrix} k & \\ & k\end{pmatrix}\right)\right|\right)$$
$$\cdot\, |\phi(\varepsilon a_m k)|\, |\det(a')|^s\, |a_m|^{ms}.$$

By Fubini's theorem, it is enough to show that the following integral (obtained by exchanging order of integration) converges in a right half plane

(3.1)

$$\int_{A_{m-1}} da' \int_K dk \int_{\mathcal{N}^-} dX \left(\delta_B^{-1}(a')\left|W\left(w_{m,m}\begin{pmatrix} a' & \\ & a'\end{pmatrix}\begin{pmatrix} I_m & a'^{-1}Xa' \\ & I_m\end{pmatrix}\begin{pmatrix} k & \\ & k\end{pmatrix}\right)\right|\right) |\det(a')|^s$$
$$\cdot \int_{F^*} |\phi(\varepsilon a_m k)|\, |a_m|^{ms}\, da_m.$$

We notice that for a fixed $k \in K$, $\int_{F^*} |\phi(\varepsilon a_m k)|\, |a_m|^{ms}\, da_m$ is a local zeta integral of Tate (see Theorem 3.3) and therefore converges absolutely for $\operatorname{Re}(s) > 0$. We claim that this integral is uniformly bounded on K: Since ϕ is a Schwartz function, its support is open and compact and therefore the set

$$\operatorname{supp}\phi \cdot K = \{x \cdot k \mid x \in \operatorname{supp}\phi,\ k \in K\}$$

is compact, as an image of a compact set $(\operatorname{supp}\phi \times K)$ under a continuous map. This set is also open, using the fact that $\operatorname{supp}\phi$ is open and that multiplication by an invertible matrix is a homeomorphism. Therefore the indicator function $1\chi_{\operatorname{supp}\phi\cdot K}$ is a Schwartz function.

Since ϕ is a Schwartz function, it is bounded, i.e. there exists $M > 0$ such that $|\phi(x)| \le M$, for every $x \in F^m$.

It is clear that $|\phi(x)| \le M \cdot 1\chi_{\operatorname{supp}\phi\cdot K}(x)$, for every $x \in F^m$, and therefore

$$\int_{F^*} |\phi(\varepsilon a_m k)|\, |a_m|^{ms}\, da_m \le M \cdot \int_{F^*} 1\chi_{\operatorname{supp}\phi\cdot K}(\varepsilon a_m k)\, |a_m|^{ms}\, da_m = M \cdot \int_{F^*} 1\chi_{\operatorname{supp}\phi\cdot K}(\varepsilon a_m)\, |a_m|^{ms}\, da_m.$$

The right hand side converges for every $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$ as a local zeta integral of Tate (see Theorem 3.3). The right hand side also does not depend on $k \in K$ and therefore $\int_{F^*} |\phi(\varepsilon a_m k)|\, |a_m|^{ms}\, da_m$ is uniformly bounded for $k \in K$, i.e. for every $k \in K$, we have $\int_{F^*} |\phi(\varepsilon a_m k)|\, |a_m|^{ms}\, da_m \le C(\phi, s)$, where $C(\phi, s)$ is a positive constant depending on ϕ and s only.

We are left with the integral

$$\int_{A_{m-1}} da' \int_K dk \int_{\mathcal{N}^-} dX \left(\delta_B^{-1}(a') \left| W \left(w_{m,m} \begin{pmatrix} a' & \\ & a' \end{pmatrix} \begin{pmatrix} I_m & a'^{-1}Xa' \\ & I_m \end{pmatrix} \begin{pmatrix} k & \\ & k \end{pmatrix} \right) \right| \right) |\det(a')|^s .$$

We substitute $a'^{-1}Xa' = Z$, $dX = \delta_B^{-1}(a')\, dZ$

$$\int_{A_{m-1}} da' \int_K dk \int_{\mathcal{N}^-} dZ \left(\delta_B^{-2}(a') \left| W \left(w_{m,m} \begin{pmatrix} a' & \\ & a' \end{pmatrix} \begin{pmatrix} I_m & Z \\ & I_m \end{pmatrix} \begin{pmatrix} k & \\ & k \end{pmatrix} \right) \right| \right) |\det(a')|^s .$$

We denote the entries of $a' = \mathrm{diag}\left(a'_1, a'_2, \ldots, a'_{m-1}, 1\right)$. We compute

$$w_{m,m} \begin{pmatrix} a' & \\ & a' \end{pmatrix} w_{m,m}^{-1} = w_{m,m} \mathrm{diag}\left(a'_1, a'_2, \ldots, a'_{m-1}, 1, a'_1, a'_2, \ldots, a'_{m-1}, 1\right) w_{m,m}^{-1}$$

For $1 \le i \le m$, we have that the $i, i+m$ diagonal elements of $\left(\begin{smallmatrix} a' & \\ & a' \end{smallmatrix}\right)$, which have value a'_i for $i \neq m$ and the value 1 for $i = m$, move after conjugation to $\sigma(i) = 2i-1, \sigma(i+m) = 2i$. i.e. we get the following matrix which we denote b

$$b = w_{m,m} \begin{pmatrix} a' & \\ & a' \end{pmatrix} w_{m,m}^{-1} = \mathrm{diag}\left(a'_1, a'_1, a'_2, a'_2, \ldots, a'_{m-1}, a'_{m-1}, 1, 1\right).$$

We denote $w_{m,m} \left(\begin{smallmatrix} I_m & Z \\ & I_m \end{smallmatrix}\right) w_{m,m}^{-1} = u_Z$. We use the Iwasawa decomposition for the element u_Z: $u_Z = n_Z t_Z k_Z$ where $n_Z \in N_{2m}$, $t_Z \in A_{2m-1}$, $k_Z \in K_{2m}$, and n_Z, t_Z are smooth in Z (see Proposition 3.22). Since $b n_Z b^{-1} \in N_{2m}$, the last integral is equal to

$$\int_{A_{m-1}} da' \int_K dk \int_{\mathcal{N}^-} dZ \left(\delta_B^{-2}(a') \underbrace{\left|\psi\left(b n_Z b^{-1}\right)\right|}_{=1} \left| W \left(b t_Z k_Z w_{m,m} \begin{pmatrix} k & \\ & k \end{pmatrix} \right) \right| \right) |\det(a')|^s .$$

We now recall the asymptotic expansion of Whittaker functions (see Proposition 3.13). There exists a finite set of the form $X = X_{(C_i, r_i)_{i=1}^{2m-1}}$ such that for every $W \in \mathcal{W}(\pi, \psi)$, there exist Schwartz functions $(\phi_\xi)_{\xi \in X} \subseteq \mathcal{S}\left(F^{2m-1} \times \mathrm{GL}_{2m}(\mathcal{O})\right)$ such that

$$W(ak) = \delta_{B_{2m-1}}^{\frac{1}{2}}(a) \cdot \sum_{\xi \in X} \xi(a_1, \ldots, a_{2m-1}) \phi_\xi(a_1, \ldots, a_{2m-1}, k),$$

for $a = m(a_1, \ldots, a_{2m-1})$ and $k \in \mathrm{GL}_{2m}(\mathcal{O})$.

Denote $t_Z = \mathrm{diag}(t_1, \ldots, t_{2m})$, $b = \mathrm{diag}(b_1, \ldots, b_{2m}) = \mathrm{diag}\left(a'_1, a'_1, a'_2, a'_2, \ldots, a'_{m-1}, a'_{m-1}, 1, 1\right)$, $b t_Z = m(c_1, \ldots, c_{2m-1})$, where $c_i = \frac{b_i t_i}{b_{i+1} t_{i+1}}$.

Since a Schwartz function on a product of groups is the sum of products of Schwartz functions on each group, we can write $\phi_\xi(a_1, \ldots, a_{2m-1}, k) = \sum_{i=1}^{M} \left(\prod_{j=1}^{2m-1} \phi_\xi^{i,j}(a_j) \right) \phi_\xi^{i,K}(k)$, where $\phi_\xi^{i,j} \in \mathcal{S}(F)$ and $\phi_\xi^{i,K} \in \mathcal{S}(\mathrm{GL}_{2m}(\mathcal{O}))$. Therefore, it suffices to consider the convergence of

$$\int_{A_{m-1}} da' \int_K dk \int_{\mathcal{N}^-} |\det(a')|^s \, dZ \delta_B^{-2}(a') \delta_{B_{2m-1}}^{\frac{1}{2}}(b t_Z) \cdot$$
$$\cdot \left| \sum_{\xi \in X} \sum_{i=1}^{M} \xi(c_1, \ldots, c_{2m-1}) \left(\prod_{j=1}^{2m-1} \phi_\xi^{i,j}(c_j) \right) \phi_\xi^{i,K}\left(k_Z w_{m,m} \begin{pmatrix} k & \\ & k \end{pmatrix} \right) \right| .$$

Using the triangle inequality, it suffices to show that an integral of the following form converges:

$$\int_{A_{m-1}} da' \int_K dk \int_{\mathcal{N}^-} dZ \left|\det\left(a'\right)\right|^s dZ \delta_B^{-2}\left(a'\right) \delta_{B_{2m-1}}^{\frac{1}{2}}\left(bt_Z\right) \cdot$$
$$\cdot \left|\xi\left(c_1,\ldots,c_{2m-1}\right)\right| \prod_{j=1}^{2m-1} \left|\phi^j\left(c_j\right)\right| \left|\phi^K\left(k_Z w_{m,m}\begin{pmatrix} k & \\ & k\end{pmatrix}\right)\right|,$$

for all $W \in \mathcal{W}\left(\pi,\psi\right)$, $\xi \in X$, $\phi^j \in \mathcal{S}\left(F\right)$, $\phi^K \in \mathcal{S}\left(\mathrm{GL}_{2m}\left(\mathcal{O}\right)\right)$.

First note that since ϕ^K is a Schwartz function it is bounded, and since K is a compact set, our integral is bounded from above by

$$M \int_{A_{m-1}} da' \int_{\mathcal{N}^-} dZ \left|\det\left(a'\right)\right|^s dZ \delta_B^{-2}\left(a'\right) \delta_{B_{2m-1}}^{\frac{1}{2}}\left(bt_Z\right) \left|\xi\left(c_1,\ldots,c_{2m-1}\right)\right| \prod_{j=1}^{2m-1} \left|\phi^j\left(c_j\right)\right|,$$

for a positive constant M. Therefore, it suffices to show that this integral converges.

In order to proceed we use the following relation between $\delta_{B_{2m-1}}$ and δ_{B_m}

$$\delta_{B_{2m-1}}^{\frac{1}{2}}\left(b\right) \cdot \delta_{B_m}^{-2}\left(a'\right) = \prod_{1\le i\le m-1} \left|a_i'\right|^{-1} = \left|\det a'\right|^{-1}.$$

Thus it suffices to show that the following integral is finite

$$\int_{\mathcal{N}^-} dZ \int_{A_{m-1}} da' \left(\delta_{B_{2m-1}}^{\frac{1}{2}}\left(t_Z\right) \cdot \left|\xi\left(c_1,\ldots,c_{2m-1}\right)\right| \prod_{j=1}^{2m-1} \left|\phi^j\left(c_j\right)\right| \left|\det\left(a'\right)\right|^{s-1}\right).$$

Since $b_{2i-1} = b_{2i} = a_i'$, we have $c_{2i-1} = \frac{t_{2i-1}}{t_{2i}}$, $c_{2i} = \frac{a_i'}{a_{i+1}'}\frac{t_{2i}}{t_{2i+1}}$.

We substitute $a_i'' = \prod_{j=i}^{m-1}\left(t_{2j}\cdot t_{2j+1}^{-1}\right)\cdot a_i'$, where $a_i'' \in F^\times$, to get $c_{2i} = \frac{a_i''}{a_{i+1}''}$, $\det\left(a'\right) = \det\left(a''\right)\cdot \prod_{j=1}^{m-1}\left|\frac{t_{2j+1}}{t_{2j}}\right|^j$.

We also write $\xi\left(c_1,\ldots,c_{2m-1}\right) = \prod_{j=1}^{2m-1}\chi_j\left(c_j\right)\log^{k_j}\left|c_j\right|$, where $\chi_j : F^* \to \mathbb{C}^*$ are characters, and $0 \le k_j \in \mathbb{Z}$, and therefore we are left with the integral

$$\int_{\mathcal{N}^-} dZ \int_{A_{m-1}} da'' \left(\delta_{B_{2m-1}}^{\frac{1}{2}}\left(t_Z\right) \prod_{j=1}^{m} \left|\frac{t_{2j-1}}{t_{2j}}\right|^{\Re(\chi_{2j-1})} \left|\phi^{2j-1}\left(\frac{t_{2j-1}}{t_{2j}}\right)\right| \left|\log^{k_{2j-1}}\left|\frac{t_{2j-1}}{t_{2j}}\right|\right| \cdot \prod_{j=1}^{m-1}\left|\frac{t_{2j+1}}{t_{2j}}\right|^{j(s-1)}\right)\cdot$$
$$\cdot \prod_{j=1}^{m-1}\left|\phi^{2j}\left(\frac{a_j''}{a_{j+1}''}\right)\right| \left|\frac{a_j''}{a_{j+1}''}\right|^{\Re(\chi_{2j})} \left|\log^{k_{2j}}\left|\frac{a_j''}{a_{j+1}''}\right|\right| \left|\det\left(a''\right)\right|^{s-1}.$$

By Fubini's theorem, it suffices to show that the following integrals converge

$$\int_{\mathcal{N}^-} \delta_{B_{2m-1}}^{\frac{1}{2}}\left(t_Z\right) \prod_{j=1}^{m} \left|\frac{t_{2j-1}}{t_{2j}}\right|^{\Re(\chi_{2j-1})} \left|\phi^{2j-1}\left(\frac{t_{2j-1}}{t_{2j}}\right)\right| \left|\log^{k_{2j-1}}\left|\frac{t_{2j-1}}{t_{2j}}\right|\right| \cdot \prod_{j=1}^{m-1}\left|\frac{t_{2j+1}}{t_{2j}}\right|^{j(s-1)} dZ,$$

$$\int_{A_{m-1}} \prod_{j=1}^{m-1}\left|\phi^{2j}\left(\frac{a_j''}{a_{j+1}''}\right)\right| \left|\frac{a_j''}{a_{j+1}''}\right|^{\Re(\chi_{2j})} \left|\log^{k_{2j}}\left|\frac{a_j''}{a_{j+1}''}\right|\right| \left|\det\left(a''\right)\right|^{s-1} da''.$$

Regarding the first integral, Denote $\Phi_1(x_1,\ldots,x_m)=\prod_{j=1}^m|\phi^{2j-1}(x_j)|$. Φ_1 has a compact support and therefore there exists $R>1$ such that if $\left|\frac{t_{2i-1}}{t_{2i}}\right|>R$ for some i, then $\Phi_1\left(\frac{t_1}{t_2},\frac{t_3}{t_4},\ldots,\frac{t_{n-1}}{t_n}\right)=0$.

Consider the function $\mu_s:(F^*)^{2m}\to\mathbb{C}^*$ defined as

$$\mu_s(u_1,\ldots,u_{2m})=\prod_{1\le i<j\le 2m-1}\left|\frac{u_j}{u_i}\right|^{\frac{1}{2}}\cdot\prod_{j=1}^{m-1}\left|\frac{u_{2j+1}}{u_{2j}}\right|^{j(s-1)}\cdot\prod_{j=1}^{m}\left|\frac{u_{2j-1}}{u_{2j}}\right|^{\Re(\chi_{2j-1})}\left|\log^{k_{2j-1}}\left|\frac{u_{2j-1}}{u_{2j}}\right|\right|.$$

This function is smooth as a product of such. It is therefore bounded on the compact set $\left\{(u_1,\ldots,u_{2m})\mid\frac{1}{R}\le|u_i|\le R\right\}$, i.e. there exists $M_1>0$, such that $\mu_s(u_1,\ldots,u_{2m})\le M_1$, whenever $\frac{1}{R}\le|u_i|\le R$, for every $1\le i\le n$.

Φ_1 is a Schwartz function and therefore it is bounded, i.e. there exists $M_2>0$, such that $\Phi_1(x)\le M_2$, for every $x\in F^m$. We now claim that for every $Z\in\mathcal{N}^-$ we have the inequality

$$\Phi_1\left(\frac{t_1}{t_2},\frac{t_3}{t_4},\ldots,\frac{t_{2m-1}}{t_{2m}}\right)\mu_s(t_1,\ldots,t_{2m})\le M_1M_2\cdot 1\chi_{\left\{Z'\mid\|Z'\|\le R^{2m^2}\right\}}(Z).$$

If $\left|\frac{t_{2i-1}}{t_{2i}}\right|>R$ for some i then we have 0 on the left hand side and therefore the inequality is trivial.

If for every $1\le i\le m$, $\left|\frac{t_{2i-1}}{t_{2i}}\right|\le R$ then, by Theorem 3.15, $1\le|t_{2i-1}|\le R|t_{2i}|\le R$ and therefore $\frac{1}{R}\le|t_{2i}|\le 1$. From the inequality $\max(1,\|Z\|)^{\frac{1}{2m}}\le\prod_{\substack{1\le k\le 2m\\ k\text{ is odd}}}|t_k|$ (Proposition 3.21), we have $\|Z\|\le R^{2m^2}$. Therefore $1\chi_{\left\{Z'\mid\|Z'\|\le R^{2m^2}\right\}}(Z)=1$. Since we have that $\frac{1}{R}\le|t_{2i}|,|t_{2i-1}|\le R$, we have $\mu_s(t_1,t_2,\ldots,t_{2m-1})\le M_1$, and since $\Phi_1(x)\le M_2$, for every $x\in F^m$, we have

$$\Phi_1\left(\frac{t_1}{t_2},\frac{t_3}{t_4},\ldots,\frac{t_{2m-1}}{t_{2m}}\right)\mu_s(t_1,\ldots,t_{2m})\le M_1M_2=M_1M_2\cdot 1\chi_{\left\{Z'\mid\|Z'\|\le R^{2m^2}\right\}}(Z).$$

Since $\mathcal{N}^-\subseteq M_m(F)$ is closed, the set $\left\{Z'\in\mathcal{N}^-\mid\|Z'\|\le R^{2m^2}\right\}$ is compact as an intersection of a closed subset and a compact subset of $M_m(F)$, and therefore

$$\int_{\mathcal{N}^-}\Phi_1\left(\frac{t_1}{t_2},\frac{t_3}{t_4},\ldots,\frac{t_{2m-1}}{t_{2m}}\right)\mu_s(t_1,\ldots,t_{2m})\,dZ\le M_1M_2\int_{\mathcal{N}^-}1\chi_{\left\{Z'\mid\|Z'\|\le R^{2m^2}\right\}}(Z)\,dZ,$$

and the right hand side is finite.

Regarding the second integral, substituting $a''_i=\prod_{j=i}^{m-1}a'''_j$ yields

$$\int_{A_{m-1}}\prod_{j=1}^{m-1}\left|\phi^{2j}\left(a'''_j\right)\right|\left|\log^{k_{2j}}\left|a'''_j\right|\right|\left|a'''_j\right|^{\Re(\chi_{2j})+j(s-1)}\,da'''. \tag{3.2}$$

This integral converges as a multiple local zeta integral of Tate (see Theorem 3.3) for s, such that $\Re(\chi_{2j})+j(s-1)>0$, for every j, i.e. $s>\max\left(1-\frac{\Re(\chi_{2j})}{j}\right)_{j=1}^{m-1}$.

To conclude, we get that $J_{\pi,\psi}(z,W,\phi)$ converges, for every z with $\mathrm{Re}(z)>r_{\pi,\wedge^2}$ where

$$r_{\pi,\wedge^2}=\max\left(\{0\}\cup\left\{1-\frac{\Re(\chi)}{j}\mid 1\le j\le m-1\mid\chi\in C_{2j}\right\}\right).$$

This constant depends on the representation π only. $\square$

Remark 3.24. Using Remark 3.14, we get that if π is unitary, then $\Re(\chi) > 0$, for every $\chi \in C_j$ for every j, and therefore $0 < r_{\pi,\wedge^2} < 1$.

Remark 3.25. When π is unitary and supercuspidal, we have that $W(ak) = f(a_1, \dots, a_{2m-1}, k)$ for $f \in \mathcal{S}\left((F^*)^{2m-1} \times \mathrm{GL}_{2m}(\mathcal{O})\right)$, and $a = m(a_1, \dots, a_{2m-1})$, $k \in \mathrm{GL}_{2m}(\mathcal{O})$. This implies that the Schwartz functions ϕ^{2j} can be chosen to vanish at zero, and therefore the multiple Tate integral (3.2) converges for any s. The only integral to consider in this case is (3.1), which converges for $s > 0$. Moreover, if π is supercuspidal (not necessarily unitary), and if $\phi(0) = 0$, then (3.1) converges for all s. We obtain by using the same arguments as in the beginning of the proof the following corollary:

Corollary 3.26. *If π is supercuspidal, then $J_{\pi,\psi}(s, W, \phi)$ converges absolutely, for every $\mathrm{Re}(s) > -\frac{\Re(\omega_\pi)}{m}$, $W \in \mathcal{W}(\pi, \psi)$, $\phi \in \mathcal{S}(F^m)$. Furthermore if $\phi(0) = 0$, then $J_{\pi,\psi}(s, W, \phi)$ converges absolutely, for every $s \in \mathbb{C}$ and $W \in \mathcal{W}(\pi, \psi)$.*

Remark 3.27. Following the steps of the proof and using the observations of the previous remark we also get the following proposition:

Proposition 3.28. *If π is supercuspidal (not necessarily unitary), then for every $s \in \mathbb{C}$, the integral*

$$\int_{A_{m-1}} da' \int_K dk \int_{\mathcal{B}\backslash M} dX \left(\delta_B^{-1}(a') W\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} a'k & \\ & a'k \end{pmatrix}\right)\psi(-\mathrm{tr}X)\right)|\det(a')|^s$$

converges absolutely.

3.3. **Non-vanishing.** Let π be an irreducible unitary generic representation of $\mathrm{GL}_{2m}(F)$ and let $r_{\pi,\wedge^2} \in \mathbb{R}$ such that $J_{\pi,\psi}(s, W, \phi)$ converges for every $W \in \mathcal{W}(\pi, \psi)$, $\phi \in \mathcal{S}(F^m)$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > r_{\pi,\wedge^2}$ (See Theorem 3.23). In this subsection we show that for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > r_{\pi,\wedge^2}$, the bilinear map $(W, \phi) \mapsto J_{\pi,\psi}(s, W, \phi)$ isn't the zero map.

We begin with a recursive expression for the Haar measure on the quotient space ${}_{N_n}\backslash^{\mathrm{GL}_n(F)}$.

3.3.1. *A recursive expression for the Haar measure on ${}_{N_n}\backslash^{G_n}$.* We give an expression for the Haar measure on ${}_{N_n}\backslash^{G_n}$ using the Haar measure on ${}_{N_{n-1}}\backslash^{G_{n-1}}$, where $G_n = \mathrm{GL}_n(F)$. Here $K = \mathrm{GL}_n(\mathcal{O})$ and $Z = Z(G_n)$ is the center of G_n. The proofs are omitted.

Proposition 3.29. *For a smooth $f : {}_{N_n}\backslash^{G_n} \to \mathbb{C}$, the following holds*

$$\int_{{}_{N_n}\backslash^{G_n}} f(g)\,dg = \int_{{}_{N_{n-1}}\backslash^{G_{n-1}}} \int_Z \int_K \frac{1}{|\det g|} f\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} zk\right) dkdzdg.$$

Let $\nu_n : G_n \to {}_{N_n}\backslash^{G_n}$ be the quotient map. We give another expression for the previous integral in the special case where $\mathrm{supp}f \subseteq \nu_n(P_n \cdot K_r)$ where $K_r \subseteq K$ is a congruence subgroup, i.e. $K_r = I_n + \varpi^r M_n(\mathcal{O})$.

Proposition 3.30. *Suppose that $f : {}_{N_n}\backslash^{G_n} \to \mathbb{C}$ is a smooth function and suppose that $\mathrm{supp}f \subseteq \nu_n(P_n \cdot K_r)$ where $K_r \subseteq K$. Then there exists a positive constant $C_{K_r} > 0$ (depending on K_r only) such that*

$$\int_{{}_{N_n}\backslash^{G_n}} f(g)\,dg = C_{K_r} \int_{{}_{N_{n-1}}\backslash^{G_{n-1}}} \int_{K_r} \frac{1}{|\det g|} f\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} k\right) dkdg.$$

3.3.2. *Proof of non-vanishing.*

Theorem 3.31. *There exist a Schwartz function $\phi \in \mathcal{S}(F^m)$ and a Whittaker function $W \in \mathcal{W}(\pi,\psi)$, such that $J_{\pi,\psi}(s,W,\phi) = 1$ for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > r_{\pi,\wedge^2}$.*

We follow [JS90, Section 7, Proposition 3].

Proof. Let W be an arbitrary Whittaker function and let $K_{m,W}$ be a congruence subgroup of $K = \mathrm{GL}_m(\mathcal{O})$ such that W is invariant to right translations of elements of the form $\begin{pmatrix} k_0 & \\ & k_0 \end{pmatrix}$ where $k_0 \in K_{m,W}$. Let $\phi : F^m \to \mathbb{C}$ be the indicator function of the set $\varepsilon_m \cdot K_{m,W}$. The set $\varepsilon_m \cdot K_{m,W}$ consists of the last row of elements of $K_{m,W}$. $\varepsilon_m \cdot K_{m,W}$ is an open compact set as $K_{m,W}$ is an open-compact subset of $M_m(F)$ and the projection maps $X \mapsto X_{ij}$ are continuous and open. Therefore ϕ is a Schwartz function on F^m. Since P_m is the stabilizer of ε_m under the right action of P_m, it is clear that the integrand of $J_{\pi,\psi}(s,W,\phi)$ has support (in the variable g) which is contained in a subset of $\nu_m(P_m K_{m,W})$ (where $\nu_m : G_m \to {}_{N_m}\backslash^{G_m}$ is the quotient map). By Proposition 3.30

$$J_{\pi,\psi}(s,W,\phi) = C'_m \cdot \int_{N_{m-1}\backslash G_{m-1}} \int_{\mathcal{N}_m^-} W\left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \left(\begin{array}{cc|c} g & 0 & \\ 0 & 1 & 0_m \\ \hline & & g \quad 0 \\ 0_m & & 0 \quad 1 \end{array} \right) \right) \underbrace{\psi(-\mathrm{tr}(X))}_{=1} dX \cdot$$

$$\cdot \left|\det g\right|^{s-1} dg,$$

where C'_m is a positive constant (which equals $C_{K_{m,W}} \cdot \mu_{K_{m,W}}(K_{m,W})$).

Denote for $0 \le k \le m-1$

$$I_k(s,W) = \int_{N_k\backslash G_k} \left|\det g\right|^{s-1+2(k+1-m)}$$

$$\int_{\mathcal{N}_{k+1}^-} W\left(w_{m,m} \begin{pmatrix} I_{k+1} & 0 & X & 0 \\ 0 & I_{m-k-1} & 0 & 0 \\ 0 & 0 & I_{k+1} & 0 \\ 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \left(\begin{array}{cc|cc} g & 0 & \multicolumn{2}{c}{0_m} \\ 0 & I_{m-k} & & \\ \hline \multicolumn{2}{c|}{0_m} & g & 0 \\ & & 0 & I_{m-k} \end{array} \right) \right) dX dg.$$

Multiplying by a suitable constant, we get that $I_{m-1}(s,W) = J_{\pi,\psi}(s,W,\phi)$ for some Schwartz function ϕ.

We give a recursive expression for I_k.

We first write $X = \left(\begin{array}{c|c} Z & 0_{k\times 1} \\ \hline Y & 0 \end{array} \right)$ where $Z \in \mathcal{N}_k^-$ is a lower triangular nilpotent $k \times k$ matrix, and $y \in F^{1\times k}$. The integral becomes

$$\int_{N_k\backslash G_k} dg \int_{\mathcal{N}_k^-} dZ \int_{F^{1\times k}} dY \left|\det g\right|^{s-1+2(k+1-m)}$$

$$\cdot W\left(w_{m,m} \begin{pmatrix} I_k & 0 & 0 & Z & 0 & 0 \\ 0 & 1 & 0 & Y & 0 & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \left(\begin{array}{cc|cc} g & 0 & \multicolumn{2}{c}{0_m} \\ 0 & I_{m-k} & & \\ \hline \multicolumn{2}{c|}{0_m} & g & 0 \\ & & 0 & I_{m-k} \end{array} \right) \right).$$

We conjugate by the matrix $\left(\begin{array}{cc|cc} g & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0_m}} \\ 0 & I_{m-k} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0_m}} & g & 0 \\ & & 0 & I_{m-k} \end{array}\right)$ and substitute $Yg = Y'$, $dY' = dY \cdot |\det g|$ to get

(3.3)

$$I_k(s,W) = \int_{N_k\backslash G_k} dg \int_{\mathcal{N}_k^-} dZ \int_{F^{1\times k}} dY' \, |\det g|^{s+2(k-m)} \, W\left(w_{m,m} \begin{pmatrix} I_k & 0 & 0 & Z & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \cdot \left(\begin{array}{cc|cc} g & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0_m}} \\ 0 & I_{m-k} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0_m}} & g & 0 \\ & & 0 & I_{m-k} \end{array}\right) \begin{pmatrix} I_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & Y' & 0 & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \right).$$

For an arbitrary Whittaker function $W \in \mathcal{W}(\pi,\psi)$ and an arbitrary Schwartz function $\Phi \in \mathcal{S}\left(F^{k\times 1}\right)$, we define $W_{k,\Phi}$ as the function

(3.4)
$$W_{k,\Phi}(g) = \int_{F^{k\times 1}} W\left(g \begin{pmatrix} I_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & u & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \right) \Phi(u)\, du.$$

Since Φ has compact support, this is an integral of a Schwartz function. It results in a Whittaker function, as a linear combination of right translations of W.

We now compute $I_k(s,W')$, where $W' = W_{k,\Phi}$ for arbitrary $W \in \mathcal{W}(\pi,\psi)$, and $\Phi \in \mathcal{S}\left(F^{k\times 1}\right)$.

After substituting (3.4) in (3.3) and computing several conjugations, we get the following expression for $I_k(s,W')$:

$$\int_{N_k\backslash G_k} dg \int_{\mathcal{N}_k^-} dZ \int_{F^{1\times k}} dY \int_{F^{k\times 1}} du \left|\det g\right|^{s+2(k-m)} \Phi(u)$$

$$\cdot W\left(w_{m,m} \begin{pmatrix} I_k & 0 & 0 & 0 & Zgu & 0 \\ 0 & 1 & 0 & 0 & Yu & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & gu & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \begin{pmatrix} I_k & 0 & 0 & Z & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \right.$$

$$\left. \cdot \left(\begin{array}{c|c} \begin{matrix} g & 0 \\ 0 & I_{m-k} \end{matrix} & 0_m \\ \hline 0_m & \begin{matrix} g & 0 \\ 0 & I_{m-k} \end{matrix} \end{array} \right) \begin{pmatrix} I_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & Y & 0 & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \right).$$

Denote $M = \begin{pmatrix} I_k & 0 & 0 & 0 & Zgu & 0 \\ 0 & 1 & 0 & 0 & Yu & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & gu & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix}$. We compute the conjugation $w_{m,m} M w_{m,m}^{-1}$.

As usual, $\left(w_{m,m} M w_{m,m}^{-1}\right)_{ij} = M_{\sigma^{-1}(i),\sigma^{-1}(j)}$. The diagonal is preserved under conjugation, and the only non-diagonal elements we need to consider are those with $\left(\sigma^{-1}(i), \sigma^{-1}(j)\right) = (i', m+k+1)$ where $1 \le i' \le k+1$ or $m+1 \le i' \le m+k$ i.e.

$$j = \sigma(m+k+1) = 2(k+1),$$
$$i = \begin{cases} 2r-1 & i' = r \\ 2r & i' = r+k \end{cases},$$

where $1 \le r \le k+1$, and therefore $i \le 2r \le 2(k+1) = j$. Therefore, the conjugation is an upper triangular unipotent matrix. The only possible non-zero element above its diagonal is the element having $j = i+1$, i.e. $2(k+1) = i+1$, i.e. $i = 2k+1 = \sigma(k+1) \iff i' = k+1$. Therefore this element is $M_{k+1,m+k+1} = Yu$. Therefore, $W\left(w_{m,m} M w_{m,m}^{-1} g\right) = \psi(Yu) W(g)$, for any $g \in \mathrm{GL}_{2m}(F)$. Thus, the integration by u results in exchanging the function $\Phi(u)$

with its Fourier transform $\hat{\Phi}$ at the point Y, and we get the following expression for $I_k(s, W')$:

$$\int_{N_k\backslash G_k} dg \int_{\mathcal{N}_k^-} dZ \int_{F^{1\times k}} dY \left|\det g\right|^{s+2(k-m)} \hat{\Phi}(Y)$$

$$\cdot W\left(w_{m,m} \begin{pmatrix} I_k & 0 & 0 & Z & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \left(\begin{array}{cc|cc} g & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0_m}} \\ 0 & I_{m-k} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0_m}} & g & 0 \\ & & 0 & I_{m-k} \end{array} \right) \begin{pmatrix} I_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & Y & 0 & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \right).$$

Since the Fourier transform is a bijection between the space of Schwartz function to itself, we can choose $\hat{\Phi}$ to be any arbitrary Schwartz function. Let $\Phi_{k,W}$ be a Schwartz function such that $\widehat{\Phi_{k,W}}$ equals to the indicator function of an open compact subset $U_{k,W} \subseteq F^{1\times k}$, such that for every $y \in U_{k,W}$ and $g \in \mathrm{GL}_{2m}(F)$

$$W\left(g \begin{pmatrix} I_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & Y & 0 & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \right) = W(g).$$

Therefore, we have

$$I_k\left(s, W_{k,\Phi_{k,W}}\right) = C \cdot \int_{N_k\backslash G_k} dg \int_{\mathcal{N}_k^-} dZ \left|\det g\right|^{s+2(k-m)} \cdot$$

$$\cdot W\left(w_{m,m} \begin{pmatrix} I_k & 0 & 0 & Z & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \left(\begin{array}{cc|cc} g & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0_m}} \\ 0 & I_{m-k} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0_m}} & g & 0 \\ & & 0 & I_{m-k} \end{array} \right) \right), \tag{3.5}$$

where $C = \mu_{F^{1\times k}}(U_{k,W})$ is a positive constant. We denote for an arbitrary Whittaker function $W \in \mathcal{W}(\pi, \psi)$, $W_{(k)} = W_{k,\Phi_{k,W}}$ and $C_k(W) = \mu_{F^{1\times k}}(U_{k,W}) > 0$, where $U_{k,W}$ is an arbitrary open compact set as above and $\widehat{\Phi_{k,W}} = 1\chi_{U_k}$.

Next we define for arbitrary $W \in \mathcal{W}(\pi, \psi)$ and $\Psi \in \mathcal{S}(F^{k\times 1})$, $W^{k,\Psi}$ as the function

$$(3.6) \qquad W^{k,\Psi}(g) = \int_{F^{k\times 1}} W\left(g \begin{pmatrix} I_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & u & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \right) \Psi(u)\, du.$$

As before, this is a Whittaker function, as a finite linear combination of right translations of the Whittaker function W. We now compute $I_k(s, W'')$ where $W'' = \left(W^{k,\Psi}\right)_{(k)}$. After substituting (3.6) in (3.5) and computing several conjugations, we get the following expression for $I_k(s, W'')$:

$$(3.7) \qquad C_k\left(W^{k,\Psi}\right) \cdot \int_{N_k \backslash G_k} dg \int_{\mathcal{N}_k^-} dZ \int_{F^{k\times 1}} du\, |\det g|^{s+2(k-m)}\, \Psi(u)$$

$$\cdot W\left(w_{m,m} \begin{pmatrix} I_k & Zgu & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & gu & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \right.$$

$$\left. \begin{pmatrix} I_k & 0 & 0 & Z & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix} \left(\begin{array}{c|c} \begin{matrix} g & 0 \\ 0 & I_{m-k} \end{matrix} & 0_m \\ \hline 0_m & \begin{matrix} g & 0 \\ 0 & I_{m-k} \end{matrix} \end{array} \right) \right).$$

We denote $M' = \begin{pmatrix} I_k & Zgu & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0 \\ 0 & gu & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1} \end{pmatrix}$ and compute the conjugation $w_{m,m} M' w_{m,m}^{-1}$.

We have $\left(w_{m,m} M' w_{m,m}^{-1}\right)_{i,j} = M'_{\sigma^{-1}(i), \sigma^{-1}(j)}$. Again, the diagonal is preserved under conjugation. The only possible non-diagonal non-zero elements to consider are those with $\sigma^{-1}(j) = k+1$, $\sigma^{-1}(i) = i'$, where $1 \le i' \le k$ or $m+1 \le i' \le m+k$, i.e.

$$j = \sigma(k+1) = 2(k+1) - 1 = 2k+1,$$

$$i = \begin{cases} \sigma(r) = 2r - 1 & i' = r \\ \sigma(r+m) = 2r & i' = m + r \end{cases},$$

where $1 \le r \le k$. Therefore $i \le 2r \le 2k < 2k+1 = j$, which implies that $w_{m,m} M' w_{m,m}^{-1}$ is an upper triangular unipotent matrix. We compute its elements above the diagonal: the only possible non-zero element is the one having an index $j = 2k+1$, $i = j-1 = 2k$, and therefore its value is $M'_{\sigma^{-1}(2k), \sigma^{-1}(2k+1)} = M'_{k+m,k+1}$, which equals the last component of gu, which is equal to $\varepsilon_k g u$ (where $\varepsilon_k \in F^{1\times k}$ is the row vector having 1 in its kth position and 0 elsewhere). Therefore for every $g \in \mathrm{GL}_{2m}(F)$, we have $W\left(w_{m,m} M w_{m,m}^{-1} g\right) = \psi(\varepsilon_k g u) W(g)$. Applying this to (3.7), results in omitting the integration by u in exchange of replacing Ψ with its Fourier transform at the point $\varepsilon_k g$. We get the following expression for $I_k(s, W'')$

$$C_k\left(W^{k,\Psi}\right) \cdot \int_{N_k \backslash G_k} dg \int_{\mathcal{N}_k^-} dZ \int_{F^{k\times 1}} du \left|\det g\right|^{s+2(k-m)} \hat{\Psi}(\varepsilon_k g)$$
$$\cdot W\left(w_{m,m}\begin{pmatrix} I_k & 0 & 0 & Z & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0\\ 0 & 0 & 0 & I_k & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1}\end{pmatrix}\left(\begin{array}{cc|cc} g & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0_m}}\\ 0 & I_{m-k} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0_m}} & g & 0\\ & & 0 & I_{m-k}\end{array}\right)\right). \tag{3.8}$$

As before, since the Fourier transform is a bijection from the space of Schwartz functions to itself, we can replace Ψ with any Schwartz function on F^k. Let $K_{k,W}$ be a congruence subgroup of G_k such that $W\left(g\left(\begin{array}{cc|cc} k_0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0_m}}\\ 0 & I_{m-k} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0_m}} & k_0 & 0\\ & & 0 & I_{m-k}\end{array}\right)\right) = W(g)$, for every $g \in G_k$ and $k_0 \in K_{k,W}$. As before, the set $\varepsilon_k \cdot K_{k,W}$ is an open compact subset of $F^{1\times k}$. Let $\Psi_{k,W}$ be a Schwartz function, such that $\widehat{\Psi_{k,W}} = 1\chi_{\varepsilon_k \cdot K_{k,W}}$. Since P_k is the stabilizer of ε_k with respect to the right action of G_k, we have that integrand of $I_k(s, W'')$ has support contained in $\nu_k(P_k K_{k,W})$ (where $\nu_k : G_k \to {}_{N_k}\backslash^{G_k}$ is the quotient map). Denote $W^{(k)} = W^{k,\Psi_{k,W}}$. Applying Proposition 3.30 to (3.8) we get that there exists a positive constant C_k' such that

$$I_k\left(s, \left(W^{(k)}\right)_{(k)}\right) = C_k' \cdot C_k\left(W^{k,\Psi}\right) \cdot \int_{N_{k-1}\backslash G_{k-1}} dg \int_{\mathcal{N}_k^-} dZ \int_{F^{k\times 1}} du \left|\det g\right|^{s+2(k-m)} \Psi(u)$$
$$\cdot W\left(w_{m,m}\begin{pmatrix} I_k & 0 & 0 & Z & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & I_{m-k-1} & 0 & 0 & 0\\ 0 & 0 & 0 & I_k & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & I_{m-k-1}\end{pmatrix}\right).$$

Therefore we proved that $I_k\left(s, \left(W^{(k)}\right)_{(k)}\right) = C_k''(W) \cdot I_{k-1}(s, W)$ where $C_k''(W)$ is a positive constant depending on W.

Note that $I_0(s, W) = W(w_{m,m})$. Since π is irreducible, there exists $W \in \mathcal{W}(\pi, \psi)$ with $W(w_{m,m}) \neq 0$, and by multiplication by a suitable constant, we can assume $W(w_{m,m}) = 1$. We define a sequence of Whittaker functions $(W_k)_{k=0}^{m-1}$ by $W_0 \in \mathcal{W}(\pi, \psi)$ with $W_0(w_{m,m}) = 1$, and $W_k = \frac{1}{C_k''(W_{k-1})}\left(W_{k-1}^{(k)}\right)_{(k)}$, for $1 \le k \le m$. Then $I_k(s, W_k) = I_{k-1}(s, W_{k-1})$, and therefore $I_{m-1}(s, W_{m-1}) = I_0(s, W_0) = 1$.

As seen in the beginning of the proof, one can choose a Schwartz function ϕ_{m-1}, such that $J_{\pi,\psi}(s, W_{m-1}, \phi_{m-1}) = I_{m-1}(s, W_{m-1})$, and therefore $J_{\pi,\psi}(s, W_{m-1}, \phi_{m-1}) = 1$, for every $s \in \mathbb{C}$ in the convergence domain. $\square$

3.4. **Rational function.** In this subsection we show that in its convergence domain, $J_{\pi,\psi}(s,W,\phi)$ is a rational function in q^{-s}, for fixed $W\in\mathcal{W}(\pi,\psi)$, $\phi\in\mathcal{S}(F^m)$.

Theorem 3.32. *For a fixed $W\in\mathcal{W}(\pi,\psi)$ and $\phi\in\mathcal{S}(F^m)$, $J_{\pi,\psi}(s,W,\phi)$ converges in its convergence domain to an element of $\mathbb{C}(q^{-s})$. Furthermore, there exists a unique polynomial $p(z)\in\mathbb{C}[z]$, with $p(0)=1$, such that*

$$I_{\pi,\psi}=\operatorname{span}_{\mathbb{C}}\left\{J_{\pi,\psi}(s,W,\phi)\mid W\in\mathcal{W}(\pi,\psi),\ \phi\in\mathcal{S}(F^m)\right\}=\frac{1}{p(q^{-s})}\mathbb{C}\left[q^s,q^{-s}\right].$$

Proof. Following the steps and the notions of the proof of Theorem 3.23, we get that $J_{\pi.s}(s,W,\phi)$ equals the sum of integrals of the form

$$\int_{F^*}da_m\int_{A_{m-1}}da'''\int_{\mathcal{N}^-}dZ\int_K dk\,\psi\left(bn_Zb^{-1}\right)$$
$$\delta_{B_{2m-1}}^{\frac{1}{2}}(t_Z)\prod_{j=1}^{m}\chi_{2j-1}\left(\frac{t_{2j-1}}{t_{2j}}\right)\phi^{2j-1}\left(\frac{t_{2j-1}}{t_{2j}}\right)\log^{k_{2j-1}}\left|\frac{t_{2j-1}}{t_{2j}}\right|\cdot\prod_{j=1}^{m-1}\left|\frac{t_{2j+1}}{t_{2j}}\right|^{j(s-1)}\cdot$$
$$\prod_{j=1}^{m-1}\phi^{2j}\left(a_j'''\right)\chi_{2j}\left(a_j'''\right)\left|a_j'''\right|^{j(s-1)}\log^{k_{2j}}\left|a_j'''\right|\cdot$$
$$\phi^K\left(k_Zw_{m,m}\begin{pmatrix}k&\\&k\end{pmatrix}\right)\phi(\varepsilon a_mk)\left|a_m\right|^{ms}\omega_\pi(a_m),$$

for some $(\phi^i)_{i=1}^{2m-1}\subseteq\mathcal{S}(F)$ and $\phi^K\in\mathcal{S}(\mathrm{GL}_{2m}(\mathcal{O}))$. We denote

$$F(Z)=\delta_{B_{2m-1}}^{\frac{1}{2}}(t_Z)\prod_{j=1}^{m}\chi_{2j-1}\left(\frac{t_{2j-1}}{t_{2j}}\right)\phi^{2j-1}\left(\frac{t_{2j-1}}{t_{2j}}\right)\log^{k_{2j-1}}\left|\frac{t_{2j-1}}{t_{2j}}\right|\cdot\prod_{j=1}^{m-1}\left|\frac{t_{2j+1}}{t_{2j}}\right|^{j(s-1)},$$
$$G(a''')=\prod_{j=1}^{m-1}\phi^{2j}\left(a_j'''\right)\chi_{2j}\left(a_j'''\right)\left|a_j'''\right|^{j(s-1)}\log^{k_{2j}}\left|a_j'''\right|,$$
$$H(a_m)=\left|a_m\right|^{ms}\omega_\pi(a_m).$$

Then this integral can be written as

$$\int_{F^*}da_m\int_{A_{m-1}}da'''\int_{\mathcal{N}^-}dZ\int_K dk\psi\left(bn_Zb^{-1}\right)F(Z)G(a''')H(a_m)\phi^K\left(k_Zw_{m,m}\begin{pmatrix}k&\\&k\end{pmatrix}\right)\phi(\varepsilon a_mk).$$

We use Fubini's theorem: we first integrate by k. Since $K=\mathrm{GL}_{2m}(\mathcal{O})$ is compact and the integrand is smooth in k, integration by k results in a linear combination of expressions of the form

$$\int_{F^*}da_m\int_{A_{m-1}}da'''\int_{\mathcal{N}^-}dZ\psi\left(bn_Zb^{-1}\right)F(Z)G(a''')H(a_m)\phi^K\left(k_Zw_{m,m}\begin{pmatrix}k_i&\\&k_i\end{pmatrix}\right)\phi(\varepsilon a_mk_i),$$

for some points $k_i\in K$. Thus it suffices to show that this expression is of the requested form. Next we integrate by Z. As seen in the proof of Theorem 3.23, Z is actually integrated on a compact set. Since t_Z and n_Z are smooth in Z, so is the expression $\psi\left(bn_Zb^{-1}\right)F(Z)$. Regarding the expression $\phi^K\left(k_Zw_{m,m}\left(\begin{smallmatrix}k_i&\\&k_i\end{smallmatrix}\right)\right)$, k_Z is continuous in Z, and ϕ^K is smooth, and therefore we get that this expression is also smooth in Z. Thus the integrand is smooth in Z, and integration by Z results in a linear combination of expressions of the form

$$\int_{F^*} da_m \int_{A_{m-1}} da''' \psi\left(bn_{Z_j}b^{-1}\right) F\left(Z_j\right) G\left(a'''\right) H\left(a_m\right) \phi^K\left(k_{Z_j} w_{m,m}\begin{pmatrix} k_i & \\ & k_i \end{pmatrix}\right) \phi\left(\varepsilon a_m k_i\right),$$

for some points $Z_j \in \mathcal{N}^-$. Note that for a fixed Z_j, $F\left(Z_j\right) \in \mathbb{C}\left[q^{-s}, q^s\right]$, and therefore we are now left with the expressions

$$\int_{F^*} H\left(a_m\right) \phi\left(\varepsilon a_m k_i\right) da_m, \tag{3.9}$$

$$\int_{A_{m-1}} \psi\left(bn_{Z_j}b^{-1}\right) G\left(a'''\right) da''', \tag{3.10}$$

where $Z_j \in \mathcal{N}^-$ and $k_i \in K$ are fixed. The integral (3.9) is clearly a local zeta integral of Tate, and therefore converges to a rational function in q^{-ms}. Regarding the integral (3.10), note that $\psi\left(bn_{Z_j}b^{-1}\right)$ is smooth in a''' (as ψ is smooth and $a''' \mapsto bn_{Z_j}b^{-1}$ is continuous), and therefore (3.10) is a multiple local zeta integral of Tate. Therefore we have that $J_{\pi,\psi}\left(s, W, \phi\right)$ converges to a rational function in q^{-s}.

Denote $I_{\pi,\psi} = \operatorname{span}_{\mathbb{C}}\left\{J_{\pi,\psi}\left(s, W, \phi\right) \mid W \in \mathcal{W}\left(\pi, \psi\right), \phi \in \mathcal{S}\left(F^m\right)\right\}$. From the equivariance properties of $J_{\pi,\psi}$ (Proposition 1.10), $I_{\pi,\psi}$ is a $\mathbb{C}\left[q^{-s}, q^s\right]$ module. The characters involved in the local zeta integrals of (3.10) are in $C = \bigcup_{i=1}^{2m-1} C_i$ (see also Proposition 3.13). The integral (3.9) results in an element of $L\left(ms, \omega_\pi\right)\mathbb{C}\left[q^{-ms}, q^{ms}\right]$. C is a finite set and we have that

$$I_{\pi,\psi} \subseteq L\left(ms, \omega_\pi\right) \prod_{j=1}^{m-1} \prod_{\chi \in C} L\left(js, \chi\right) \cdot \mathbb{C}\left[q^{-s}, q^s\right],$$

It now is clear that $I_{\pi,\psi}$ is a fractional ideal of $\mathbb{C}\left[q^{-s}, q^s\right]$. By Theorem 3.31, $1 \in I_{\pi,\psi}$. We show that this implies the existence and the uniqueness of the requested polynomial $p\left(z\right)$.

Existence: Since $\mathbb{C}\left[q^{-s}, q^s\right]$ is a principal ideal domain, there exists coprime $f, g \in \mathbb{C}\left[z\right]$, such that $I_{\pi,\psi} = \frac{f\left(q^{-s}\right)}{g\left(q^{-s}\right)}\mathbb{C}\left[q^{-s}, q^s\right]$. Since $1 \in I_{\pi,\psi}$, there exists $h \in \mathbb{C}\left[z\right]$ and an integer $M \geq 0$, such that $\frac{f\left(q^{-s}\right)}{g\left(q^{-s}\right)} h\left(q^{-s}\right) q^{Ms} = 1$, i.e. $f\left(z\right) h\left(z\right) = z^M g\left(z\right)$. Since f and g are coprime, $f \mid z^M$, and therefore $f\left(q^{-s}\right) = q^{-M_1 s}$ for an integer $M_1 \geq 0$, and therefore $I_{\pi,\psi} = \frac{1}{g\left(q^{-s}\right)}\mathbb{C}\left[q^{-s}, q^s\right]$. Writing $g\left(z\right) = a \cdot z^{M_2} p\left(z\right)$, where $a \in \mathbb{C}^*$, $0 \leq M_2 \in \mathbb{Z}$, and p is a polynomial with $p\left(0\right) = 1$, implies $I_{\pi,\psi} = \frac{1}{p\left(q^{-s}\right)}\mathbb{C}\left[q^{-s}, q^s\right]$, and p is a polynomial as requested.

Uniqueness: suppose that $I_{\pi,\psi} = \frac{1}{p_1\left(q^{-s}\right)}\mathbb{C}\left[q^{-s}, q^s\right] = \frac{1}{p_2\left(q^{-s}\right)}\mathbb{C}\left[q^{-s}, q^s\right]$. Then $p_1\left(q^{-s}\right) = r\left(q^{-s}\right) \cdot p_2\left(q^{-s}\right)$, where $r\left(z\right)$ is an invertible element of $\mathbb{C}\left[z, z^{-1}\right]$, i.e. $r\left(z\right) = a \cdot z^M$, where $a \in \mathbb{C}^*$ and $M \in \mathbb{Z}$, i.e. $p_1\left(z\right) = a \cdot z^M \cdot p_2\left(z\right)$. Since $p_1\left(0\right) = p_2\left(0\right) = 1$, this implies $a = 1$, $M = 0$. □

Remark 3.33. Suppose that π is supercuspidal. In this case, $\left(\phi^i\right)_{i=1}^{2m-1}$ can be chosen, such that $\phi^i\left(0\right) = 0$ for every i (see also Remark 3.25). This implies that the integral (3.10) results in an element of $\mathbb{C}\left[q^{-s}, q^s\right]$ and therefore $J_{\pi,\psi}\left(s, W, \phi\right)$ results in an element of $L\left(ms, \omega_\pi\right)\mathbb{C}\left[q^{-s}, q^s\right]$. Furthermore, if $\phi\left(0\right) = 0$, then the integral (3.9) results in an element of $\mathbb{C}\left[q^{-ms}, q^{ms}\right]$ and therefore in this case $J_{\pi,\psi}\left(s, W, \phi\right)$ results in an element of $\mathbb{C}\left[q^{-s}, q^s\right]$.

Remark 3.34. The calculations done in Subsection 1.2.3 show that the set $I_{\pi,\psi}$ does not depend on the choice of the character ψ (Since for $a \in F^*$, the expressions $J_{\pi,\psi}\left(s, \pi\left(\left(\begin{smallmatrix} I_m & \\ & a^{-1}I_m \end{smallmatrix}\right)\right) W, \phi\right)$ and $J_{\pi,\psi_a}(s, W^a, \phi)$ differ by multiplication by an invertible element of $\mathbb{C}[q^s, q^{-s}]$). We denote $L(s, \pi, \wedge^2) = \frac{1}{p(q^{-s})}$ where $p(z)$ is as in the theorem.

Corollary 3.35. *For every $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in \mathcal{S}(F^m)$, $J_{\pi,\psi}(s, W, \phi)$ and $\tilde{J}_{\pi,\psi}(s, W, \phi)$ have meromorphic continuations, for all $s \in \mathbb{C}$, which we continue to denote $J_{\pi,\psi}(s, W, \phi)$ and $\tilde{J}_{\pi,\psi}(s, W, \phi)$. The meromorphic continuations of $J_{\pi,\psi}(s, W, \phi)$ and $\tilde{J}_{\pi,\psi}(s, W, \phi)$ have the same equivariance properties as the original forms.*

Proof. Since we have shown that $J_{\pi,\psi}(s, W, \phi)$ has a meromorphic continuation, so does $\tilde{J}_{\pi,\psi}(s, W, \phi)$ (as it is defined using $J_{\tilde{\pi},\psi^{-1}}$). For every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > r_{\pi,\wedge^2}$, we have (Proposition 1.10)

$$J_{\pi,\psi}\left(s, \pi\left(\begin{pmatrix} g & X \\ & g \end{pmatrix}\right) W, \rho(g)\phi\right) = |\det g|^{-s} \psi\left(\mathrm{tr}\left(g^{-1}X\right)\right) J_{\pi,\psi}(s, W, \phi),$$

and both sides of the equation are rational functions in the variable q^{-s}. By the uniqueness theorem, the equation remains valid for every $s \in \mathbb{C}$. □

3.5. **The functional equation.** Let π be an irreducible supercuspidal representation of $\mathrm{GL}_{2m}(F)$. In this subsection we prove the following

Theorem 3.36. *There exists a non-zero element $\gamma_{\pi,\psi}(s)$ of $\mathbb{C}(q^{-s})$ such that for every $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in \mathcal{S}(F^m)$ the following equation holds*

$$\tilde{J}_{\pi,\psi}(s, W, \phi) = \gamma_{\pi,\psi}(s) \cdot J_{\pi,\psi}(s, W, \phi).$$

Furthermore,

$$\gamma_{\pi,\psi}(s) = \varepsilon_{\pi,\psi}(s) \cdot \frac{L(1-s, \tilde{\pi}, \wedge^2)}{L(s, \pi, \wedge^2)},$$

where $\varepsilon_{\pi,\psi}(s)$ is an invertible element of $\mathbb{C}[q^{-s}, q^s]$.

In this subsection, we denote $G_m = \mathrm{GL}_m(F)$. We denote by P_{2m} the mirabolic subgroup of G_{2m}:

$$P_{2m} = \left\{ \begin{pmatrix} g & * \\ & 1 \end{pmatrix} \mid g \in \mathrm{GL}_{2m-1}(F) \right\}.$$

We denote by $M_{m,m}$ the Levi subgroup of G_{2m} corresponding to the partition (m, m), by $P_{m,m}$ the parabolic subgroup of G_{2m} corresponding to this partition, and by $N_{m,m}$ the unipotent radical of $P_{m,m}$.

In order to prove this functional equation, we first construct an embedding of $\mathrm{Hom}_{P_{2m}\cap S_{2m}}(\pi, \Psi)$ into $\mathrm{Hom}_{P_{2m}\cap M_{m,m}}(\pi, 1)$ and show that latter has dimension ≤ 1. We show that it follows that

$$\dim \mathrm{Hom}_{S_{2m}}\left(\pi \otimes \mathcal{S}(F^m), |\cdot|^{-s} \cdot \Psi\right) \leq 1,$$

and therefore $J_{\pi,\psi}$ and $\tilde{J}_{\pi,\psi}$ are proportional. Since $\gamma_{\pi,\psi}(s)$ is the quotient of two rational functions, it follows that $\gamma_{\pi,\psi}(s) \in \mathbb{C}(q^{-s})$. We then show that $\gamma_{\pi,\psi}$ has the requested form.

3.5.1. *Multiplicity one theorem.* In this subsection we prove the following Multiplicity one theorem:

Theorem 3.37. *Let π be a supercuspidal irreducible representation of G_{2m}. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{P_{2m}\cap M_{m,m}}(\pi, 1) \leq 1.$$

We will need some preparations in order to prove this theorem. We follow [Mat12].

Let n be a positive integer. We denote $G_n = \mathrm{GL}_n(F)$. We think of $G_k \subseteq G_n$ (for $k < n$), using the standard embedding $g \mapsto \begin{pmatrix} g & \\ & I_{n-k} \end{pmatrix}$.

Let

$$P_n = P_n(F) = \left\{ \begin{pmatrix} g & * \\ 0 & 1 \end{pmatrix} \mid g \in \mathrm{GL}_{n-1}(F) \right\}$$

be the mirabolic subgroup.

For any $0 \leq a, b$ such that $a + b \leq n$ we define

$$M_{a,b}^{(n)} = \left\{ \begin{pmatrix} g_a & & \\ & g_b & \\ & & I_{n-(a+b)} \end{pmatrix} \mid g_a \in \mathrm{GL}_a(F), g_b \in \mathrm{GL}_b(F) \right\},$$

and we denote $M_{a,b} = M_{a,b}^{(n)}$ if $a + b = n$.

Let $U_n = N_{n-1,1} = \left\{ \begin{pmatrix} I_{n-1} & v \\ & 1 \end{pmatrix} \mid v \in F^{n-1} \right\}$. Then $P_n = G_{n-1} \cdot U_n$. For a representation π of P_{n-1}, denote $\Phi^+(\pi) = \mathrm{ind}_{P_{n-1}U_n}^{P_n}(\pi')$, where

$$\pi'(p \cdot u) = (\pi \otimes \psi)(p \cdot u) = \psi(u)\pi(p),$$

for $u \in U_n$, $p \in P_{n-1}$.

Let $p \geq q \geq 0$ such that $p + q = n$. We define $\sigma_{p,q}$ as the following permutation

$$\begin{pmatrix} 1 & 2 & \ldots & p-q & p-q+1 & p-q+2 & \ldots & p-1 & p & p+1 & p+2 & \ldots & p+q \\ 1 & 2 & \ldots & p-q & p-q+1 & p-q+3 & \ldots & p+q-3 & p+q-1 & p-q+2 & p-q+4 & \ldots & p+q \end{pmatrix},$$

and $w_{p,q}$ as the column permutation matrix of $\sigma_{p,q}$.

Let $H_{p,q}^{(n)} = w_{p,q} M_{p,q} w_{p,q}^{-1}$ and let $H_{p,q-1}^{(n)} = w_{p,q} M_{p,q-1}^{(n)} w_{p,q}^{-1}$. Note that since $\sigma_{p,q}(n) = n$, and since $\left(w_{p,q} m w_{p,q}^{-1}\right)_{i,j} = m_{\sigma_{p,q}^{-1}(i), \sigma_{p,q}^{-1}(j)}$, we have that $H_{p,q-1}^{(n)} \subseteq G_{n-1}$.

We also denote

$$H_{p-1,q-1}^{(n)} = \left\{ \begin{pmatrix} h & \\ & I_2 \end{pmatrix} \mid h \in H_{p-1,q-1}^{(n-2)} \right\}.$$

Lemma 3.38. *Let $p \geq q \geq 1$ such that $p + 1 = n$, and let*

$$S_{p,q}^{(n)} = \left\{ g \in G_{n-1} \mid \psi\left(gug^{-1}\right) = 1 \, \forall u \in U_n \cap H_{p,q}^{(n)} \right\}.$$

Then $S_{p,q}^{(n)} = P_{n-1} \cdot H_{p,q-1}^{(n)}$.

Proof. Let $g = \begin{pmatrix} g_0 & \\ & 1 \end{pmatrix} \in G_{n-1}$ and let $u = \begin{pmatrix} I_{n-1} & x \\ & 1 \end{pmatrix}$ where $x \in F^{n-1}$, then

$$gug^{-1} = \begin{pmatrix} I_{n-1} & g_0 x \\ & 1 \end{pmatrix}.$$

Let $\mathrm{row}_{n-1}(g_0)$ denote the $(n-1)$th row of g_0, then $\psi(gug^{-1}) = \psi(\mathrm{row}_{n-1}(g_0) \cdot x)$.

Elements of $M_{p,q}^{(n)}$ have as their last column, a column consisting of 0 at the first p places, and therefore elements of $H_{p,q}^{(n)}$ have as their last column, a column which consists of zeros at the places $\sigma_{p,q}(1), \dots, \sigma_{p,q}(p)$. Therefore

$$U_n \cap H_{p,q}^{(n)} = \left\{ \begin{pmatrix} I_{n-1} & x \\ & 1 \end{pmatrix} \mid x \in F^{n-1} \mid x_{\sigma_{p,q}(1)} = \dots = x_{\sigma_{p,q}(p)} = 0 \right\}.$$

Therefore if $\forall u \in U_n \cap H_{p,q}^{(n)}$, we have $\psi(gug^{-1}) = 1$, then $\mathrm{row}_{n-1}(g_0)$ must have zeros at the places $\sigma_{p,q}(p+1), \dots, \sigma_{p,q}(p+q-1)$ - otherwise if $\mathrm{row}_{n-1}(g_0)$ doesn't have zero in an element $\sigma_{p,q}(p+i)$ for $1 \le i \le q-1$, we can choose an element $u = \left(\begin{smallmatrix} I_{n-1} & x \\ & 1 \end{smallmatrix}\right) \in U_n \cap H_{p,q}^{(n)}$, with x being a vector having zeros everywhere except for the $(p+i)$th place, where we can put an element such that $\mathrm{row}_{n-1}(g_0) \cdot x = a$, where $\psi(a) \neq 1$, and then $\psi(gug^{-1}) = \psi(\mathrm{row}_{n-1}(g_0) \cdot x) \neq 1$.

It is clear from the equality $\psi(gug^{-1}) = \psi(\mathrm{row}_{n-1}(g_0) \cdot x)$ and from the computation of $U_n \cap H_{p,q}^{(n)}$ that if $\mathrm{row}_{n-1}(g_0)$ consists of zeros at the places $\sigma_{p,q}(p+1), \dots, \sigma_{p,q}(p+q-1)$, then $g \in S_{p,q}^{(n)}$.

Therefore

$$S_{p,q}^{(n)} = \{ g \in G_{n-1} \mid \mathrm{row}_{n-1}(g) \text{ has zeros at the places } \sigma_{p,q}(p+1), \dots, \sigma_{p,q}(p+q-1) \}.$$

We claim that this set equals $P_{n-1} \cdot H_{p,q-1}^{(n)}$. Regarding the inclusion $P_{n-1} \cdot H_{p,q-1}^{(n)} \subseteq S_{p,q}^{(n)}$: let $p' \in P_{n-1}$, $h \in H_{p,q-1}^{(n)}$, then the $(n-1)$th row of $p'h$ equals the $(n-1)$th row of h. Write $h = w_{p,q} m w_{p,q}^{-1}$, where $m = \left(\begin{smallmatrix} g_p & \\ & g_q \end{smallmatrix}\right)$, where $g_p \in \mathrm{GL}_p(F)$, $g_q \in \mathrm{GL}_q(F)$, then $h_{ij} = m_{\sigma_{p,q}^{-1}(i), \sigma_{p,q}^{-1}(j)}$, and therefore $h_{n-1,j} = m_{p, \sigma_{p,q}^{-1}(j)}$ and

$$h_{n-1, \sigma_{p,q}(p+j)} = m_{p,p+j} = 0,$$

for every $1 \le j \le q-1$.

Regarding the inclusion $S_{p,q}^{(n)} \subseteq P_{n-1} \cdot H_{p,q-1}^{(n)}$, suppose $g = \left(\begin{smallmatrix} g_0 & \\ & 1 \end{smallmatrix}\right)$ with $g_0 \in \mathrm{GL}_{n-1}(F)$ and that $\mathrm{row}_{n-1}(g_0)$ has zeroes at the places $\sigma_{p,q}(p+1), \dots, \sigma_{p,q}(p+q-1)$. Choose any matrix $m \in M_{p,q-1}^{(n)}$, such that $m_{p,j} = (g_0)_{n-1, \sigma_{p,q}(j)}$, for $1 \le j \le p$. Then $h = w_{p,q} m w_{p,q}^{-1} \in H_{p,q-1}^{(n-1)}$, and h and g share the same $(n-1)$th row. Therefore $gh^{-1} \in P_{n-1}$ and $g \in P_{n-1} H_{p,q-1}^{(n-1)}$, as required. ∎

Lemma 3.39. *Let $p \ge q \ge 2$ such that $p+q = n$, and let*

$$S_{p,q-1}^{(n)} = \left\{ g \in G_{n-2} \mid \psi\left(gug^{-1}\right) = 1 \, \forall u \in U_{n-1} \cap H_{p,q-1}^{(n-1)} \right\}.$$

Then $S_{p,q-1}^{(n)} = P_{n-2} \cdot H_{p-1,q-1}^{(n)}$.

Remark 3.40. As noted above, $H_{p,q-1}^{(n-1)} \subseteq G_{n-1}$. We may think of all groups in the lemma as subgroups of $\mathrm{GL}_{n-1}(F)$.

Proof. Let $g = \left(\begin{smallmatrix} g_0 & \\ & 1 \end{smallmatrix}\right)$, where $g_0 \in \mathrm{GL}_{n-2}(F)$ and $u = \left(\begin{smallmatrix} I_{n-2} & x \\ & 1 \end{smallmatrix}\right)$, where $x \in F^{n-2}$. Then, as before, $gug^{-1} = \left(\begin{smallmatrix} I_{n-2} & g_0 x \\ & 1 \end{smallmatrix}\right)$. Again, $\psi(gug^{-1}) = \psi(\mathrm{row}_{n-2}(g_0) \cdot x)$, where $\mathrm{row}_{n-2}(g_0)$ denotes the $(n-2)$th row of g_0. We compute $U_{n-1} \cap H_{p,q-1}^{(n-1)}$. First, we notice that $\sigma_{p,q}(p) = p+q-1 = n-1$. Elements of $M_{p,q-1}^{(n)}$ have zeros at the pth column at positions $p+1, \dots,$

$p+q-2$, and therefore elements of $H_{p,q-1}^{(n)}$ have zeros at the $(n-1)$th column at positions $\sigma_{p,q}(p+1), \ldots, \sigma_{p,q}(p+q-2)$. Therefore we get

$$U_{n-1} \cap H_{p,q-1}^{(n-1)} = \left\{ \begin{pmatrix} I_{n-2} & x \\ & 1 \end{pmatrix} \mid x \in F^{n-2} \mid x_{\sigma_{p,q}(p+1)} = \cdots = x_{\sigma_{p,q}(p+q-2)} = 0 \right\}.$$

As before, since $\psi\left(gug^{-1}\right) = \psi\left(\mathrm{row}_{n-2}(g_0) \cdot x\right)$, we get that if $\psi\left(gug^{-1}\right) = 1$ $\forall u \in U_{n-1} \cap H_{p,q-1}^{(n-1)}$, then $\mathrm{row}_{n-2}(g_0)$ must have zeros at the places $\sigma_{p,q}(1), \sigma_{p,q}(2), \ldots, \sigma_{p,q}(p)$, and that

$$S_{p,q-1}^{(n)} = \left\{ \begin{pmatrix} g_0 & \\ & 1 \end{pmatrix} \mid \mathrm{row}_{n-2}(g_0) \text{ has zeros at the places } \sigma_{p,q}(1), \sigma_{p,q}(2), \ldots, \sigma_{p,q}(p) \right\}.$$

As before, we claim that $S_{p,q-1}^{(n)} = P_{n-2} \cdot H_{p-1,q-1}^{(n)}$. For the inclusion $S_{p,q-1}^{(n)} \supseteq P_{n-2} \cdot H_{p-1,q-1}^{(n)}$, one writes $g = p'h$ where $p' \in P_{n-2}$ and $h \in H_{p-1,q-1}^{(n)}$. Then the $(n-2)$th row of g equals the $(n-2)$th row of h. Write $h = w_{p-1,q-1} m w_{p-1,q-1}^{-1}$ where $m \in M_{p-1,q-1}^{(n-2)}$. Then $\sigma_{p-1,q-1}(n-2) = n-2 = p+q-2 > p-1$ and $h_{n-2,j} = m_{n-2,\sigma_{p-1,q-1}^{-1}(j)}$. Since $n-2 > p-1$, $m_{n-2,j} = 0$, for $1 \leq j \leq p-1$, and therefore $h_{n-2,\sigma_{p-1,q-1}(j)} = 0$, for $1 \leq j \leq p-1$. Note that for $1 \leq j \leq p-1$

$$\sigma_{p-1,q-1}(j) = \begin{cases} j & 1 \leq j \leq p-q \\ p-q+2k-1 & j = p-q+k,\ (1 \leq k \leq q-1) \end{cases} = \sigma_{p,q}(j),$$

and therefore $h_{n-2,\sigma_{p,q}(j)} = 0$, for $1 \leq j \leq p-1$. Finally, $\sigma_{p,q}(p) = n-1$ and $h_{n-2,n-1} = m_{n-2,n-1} = 0$ (as $m \in G_{n-2}$). Therefore $g \in S_{p,q-1}^{(n)}$. The other inclusion is shown as in the previous proof. □

Proposition 3.41. *Suppose $p \geq q \geq 1$ with $p+q = n$. Let (σ, V) be a representation of P_{n-1}, and let χ be a positive character of $P_n \cap H_{p,q}^{(n)}$. Then there exists a positive character χ' of $P_{n-1} \cap H_{p,q-1}^{(n)}$, such that*

$$\mathrm{Hom}_{P_n \cap H_{p,q}^{(n)}}\left(\Phi^+(\sigma), \chi\right) \hookrightarrow \mathrm{Hom}_{P_{n-1} \cap H_{p,q-1}^{(n)}}\left(\sigma, \chi'\right).$$

Proof. Denote $W = \Phi^+(V) = \Phi^+(\sigma) = \mathrm{ind}_{P_{n-1}U_n}^{P_n}(\sigma')$ where $\sigma' = \sigma \otimes \psi$ defined as above. Let A be the projection operator from $\mathcal{S}(P_n, V)$ to $W = \mathrm{ind}_{P_{n-1}U_n}^{P_n}(\sigma')$, defined as

$$(Af)(p) = \int_{P_{n-1}U_n} \sigma'^{-1}(y) f(yp)\, d\mu_{P_{n-1}U_n,r}(y).$$

Since f is a Schwartz function, for a fixed p, the integral is integrated on $(\mathrm{supp} f)\, p^{-1}$, and therefore converges. A direct computation shows that $Af \in \mathrm{ind}_{P_{n-1}U_n}^{P_n}(\sigma')$. One can show that A is surjective.

Let $L \in \mathrm{Hom}_{P_n \cap H_{p,q}^{(n)}}(\Phi^+\sigma, \chi)$. We define using A and L a distribution $T = L \circ A : \mathcal{S}(P_n, V) \to \mathbb{C}$. A direct computation shows that this distribution satisfies

$$\langle T, \rho(h_0) f \rangle = \chi(h_0) \langle T, f \rangle, \qquad \forall h_0 \in P_n \cap H_{p,q}^{(n)}, \tag{3.11}$$

$$\langle T, \lambda(y_0) f \rangle = \delta_{P_{n-1}U_n}(y_0) \left\langle T, \sigma'^{-1}(y_0) f \right\rangle, \qquad \forall y_0 \in P_{n-1}U_n. \tag{3.12}$$

Therefore the map $L \mapsto L \circ A$ defines a map from $\mathrm{Hom}_{P_n \cap H_{p,q}^{(n)}}\left(\Phi^+(\sigma), \chi\right)$ to the subspace of distributions on $\mathcal{S}(P_n, V)$ satisfying the relations (3.11) and (3.12). This map is injective, since A is surjective.

We define for $u \in U_n$ and $g \in G_{n-1}$, $\Psi(ug) = \psi(u)$. This is well defined, as if $u_1 g_1 = u_2 g_2$, then $u_2^{-1} u_1 = g_2 g_1^{-1} \in G_{n-1} \cap U_n = \{I_n\}$, and therefore $u_1 = u_2$.

We have that for $u_1, u_2 \in U_n$ and $g_2 \in G_{n-1}$

$$\Psi(u_1 u_2 g_2) = \psi(u_1 u_2) = \psi(u_1)\Psi(u_2 g_2).$$

Let T be a distribution on $\mathcal{S}(P_n, V)$ satisfying the relations (3.11) and (3.12). We define $\Psi \cdot T$ as the following distribution:

$$\langle \Psi \cdot T, f \rangle = \langle T, \Psi \cdot f \rangle.$$

One can check that for $u \in U_n$ we have

$$\langle \lambda(u)(\Psi \cdot T), f \rangle = \langle T \cdot \Psi, f \rangle.$$

To show this, one uses the fact that for $p \in P_n$ we have $\Psi(u^{-1}p) = \psi(u^{-1})\Psi(p) f(p)$ and that $\delta_{P_{n-1}U_n} \restriction_{U_n} \equiv 1$. Therefore, $\Psi \cdot T$ is left invariant to translations by U_n. It follows that there exists a distribution S on $\mathcal{S}(G_{n-1}, V)$, such that

$$\langle \Psi \cdot T, f \rangle = \int_{G_{n-1}} \left[\int_{U_n} f(ug)\, du\right] dS(g).$$

(This eventually follows from the well known fact that the averaging map $\alpha : \mathcal{S}(P_n, V) \to \mathcal{S}\left({}_{U_n}\backslash^{P_n}, V\right)$ defined by $(\alpha(f))(p) = \int_{U_n} f(up)\, du$ is surjective)

A simple computation shows that for $u_0 \in U_n$, $f \in \mathcal{S}(P_n, V)$ we have $\langle \Psi \cdot T, f \rangle = \langle \Psi \cdot T, \rho(u_0) f \rangle$.

Note that since χ is positive and $U_n \cap H_{p,q}^{(n)}$ is a subgroup of F^{n-1}, χ must be trivial on this subgroup (as for every $a \in U_n \cap H_{p,q}^{(n)}$, belongs to a compact subgroup K_a, and therefore $\chi(K_a)$ is compact, but since χ is positive, $\chi \restriction_{K_a} \equiv 1$ and therefore $\chi(a) = 1$). Therefore we get that $\langle T, \rho(u) f \rangle = \langle T, f \rangle$, for every $u \in U_n \cap H_{p,q}^{(n)}$.

Using both equalities yields

$$\langle T, \rho(u)\Psi f \rangle = \langle T, \Psi f \rangle \qquad\qquad \forall u \in U_n \cap H_{p,q}^{(n)},$$

for every $f \in \mathcal{S}(P_n)$.

This implies that for $g_0 \in \mathrm{supp} S$, we have $\Psi(g_0 u_0) = \Psi(g_0)$, which implies $\mathrm{supp} S \subseteq S_{p,q}^{(n)}$ and $\mathrm{supp} T = \mathrm{supp}(\Psi \cdot T) \subseteq U_n \cdot S_{p,q}^{(n)}$. Using the decomposition $S_{p,q}^{(n)} = P_{n-1} H_{p,q-1}^{(n)}$, we have that

$$\mathrm{supp} T \subseteq P_{n-1} U_n H_{p,q-1}^{(n)}.$$

Hence that map $T \mapsto T \restriction_{\mathcal{S}\left(P_{n-1}U_n H_{p,q-1}^{(n)}, V\right)}$ is injective.

Consider the projection $B : \mathcal{S}\left(P_{n-1}U_n \times H_{p,q-1}^{(n)}, V\right) \to \mathcal{S}\left(P_{n-1}U_n H_{p,q-1}^{(n)}, V\right)$ defined by

$$(Bf)\left(y^{-1}h\right) = \int_{P_{n-1} \cap H_{p,q-1}^{(n)}} f(ay, ah)\, d\mu_r(a) \qquad \left(y \in P_{n-1}U_n,\ h \in H_{p,q-1}^{(n)}\right).$$

This is well defined as if $y_1^{-1}h_1 = y_2^{-1}h_2$, then

$$y_1 y_2^{-1} = h_1 h_2^{-1} \in H^{(n)}_{p,q-1} \cap (P_{n-1}U_n) = P_{n-1} \cap H^{(n)}_{p,q-1}.$$

(the sets are equal, since $h = pu \implies u = p^{-1}h \in G_{n-1} \cap U_n = \{I_n\}$).

Therefore by substituting $a = a' \cdot y_2 y_1^{-1}$, we get the required equality of the integrals.

One can show that B is surjective.

Consider the isomorphism $\phi \mapsto \tilde{\phi}$ of $\mathcal{S}\left(P_{n-1}U_n \times H^{(n)}_{p,q-1}, V\right)$, defined by

$$\tilde{\phi}(y,h) = \chi(h)\,\delta_{P_{n-1}U_n}(y)\,\sigma'\left(y^{-1}\right)\phi(y,h)$$

Let T be a distribution on $\mathcal{S}(P_n, V)$ satisfying the relations as above. We define a distribution D on $\mathcal{S}\left(P_{n-1}U_n \times H^{(n)}_{p,q-1}, V\right)$ by $\langle D, \phi\rangle = \left\langle T, B\left(\tilde{\phi}\right)\right\rangle$. Let $\phi_1 = \rho(y_0, h_0)\,\phi$, for $y_0 \in P_{n-1}U_n$, $h_0 \in H^{(n)}_{p,q-1}$. A direct calculation shows that

$$\widetilde{\phi_1} = \chi(h_0)^{-1}\,\delta_{P_{n-1}U_n}(y_0)^{-1}\,\sigma'(y_0)\,\rho(y_0,h_0)\,\tilde{\phi},$$

which implies that

$$B\left(\widetilde{\phi_1}\right) = \chi(h_0)^{-1}\,\delta_{P_{n-1}U_n}(y_0)^{-1}\left(\rho(h_0)\,\lambda(y_0)\,\sigma'(y_0)\,B\left(\tilde{\phi}\right)\right),$$

which implies that $\left\langle T, B\left(\widetilde{\phi_1}\right)\right\rangle = \left\langle T, B\left(\tilde{\phi}\right)\right\rangle$.

Therefore we get that $\langle D, \rho(y_0,h_0)\,\phi\rangle = \langle D, \phi\rangle$, for any $y_0 \in P_{n-1}U_n$, $h_0 \in H^{(n)}_{p,q-1}$. This means that D is invariant to right translations of $P_{n-1}U_n \times H^{(n)}_{p,q-1}$. It follows that there exists a unique functional ξ_D on V, such that $\langle D, \phi\rangle = \int_{H^{(n)}_{p,q-1}} \int_{P_{n-1}U_n} \langle \xi_D, \phi(y,h)\rangle\, d_r(y)\, d_r(h)$ (see [War72, Proposition 5.2.1.2]).

Now let $b \in H^{(n)}_{p,q-1} \cap P_{n-1}$. Let $\phi_1 = \lambda(b,b)\,\phi$. A simple calculation yields

$$\widetilde{\phi_1} = \chi(b)\,\delta_{P_{n-1}U_n}(b)\left(\lambda(b,b)\left(\widetilde{\sigma'\left(b^{-1}\right)\phi}\right)\right).$$

Note that for an arbitrary $f \in \mathcal{S}\left(P_{n-1}U_n \times H^{(n)}_{p,q-1}, V\right)$, we have

$$B(\lambda(b,b)\,f)\left(y^{-1}h\right) = \delta_1(b)\,B(f)\left(y^{-1}h\right),$$

where $\delta_1 = \delta_{P_{n-1}\cap H^{(n)}_{p,q-1}}$. Therefore

$$\langle D, \lambda(b,b)\,\phi\rangle = \chi(b)\,\delta_{P_{n-1}U_n}(b)\,\delta_1(b)\left\langle D, \sigma'\left(b^{-1}\right)\phi\right\rangle.$$

On the other hand one has

$$\langle D, \lambda(b,b)\,\phi\rangle = \delta(b)\,\langle D, \phi\rangle,$$

where $\delta = \delta_{H^{(n)}_{p,q-1}}(b)\,\delta_{P_{n-1}U_n}(b)$, and therefore

$$\left\langle D, \sigma'\left(b^{-1}\right)\phi\right\rangle = \chi(b)^{-1}\,\delta(b)^{-1}\,\delta_1(b)\,\delta_{P_{n-1}U_n}(b)\,\langle D, \phi\rangle.$$

Denote $\chi'(b)^{-1} = \chi(b)\,\delta(b)\,\delta_1(b)^{-1}\,\delta_{P_{n-1}U_n}(b)^{-1}$. This is a positive character of $P_{n-1}\cap H^{(n)}_{p,q-1}$, as a product of such. Using the uniqueness of ξ_D we get that $\langle \sigma'(b)\,\xi_D, v\rangle = \chi'(b)^{-1}\,\langle \xi_D, v\rangle$, for every $v \in V$ and $b \in H^{(n)}_{p,q-1} \cap P_{n-1}$. This implies that $\xi_D \in \mathrm{Hom}_{H^{(n)}_{p,q-1}\cap P_{n-1}}(\sigma, \chi')$.

Therefore we get the requested embedding as the following composition of injective maps:

$$\begin{aligned}
&L \mapsto L \circ A = T, \\
&T \mapsto T \restriction_{\mathcal{S}\left(P_{n-1}U_n H^{(n)}_{p,q-1}, V\right)} = T', \\
&T' \mapsto T' \circ B \circ \tilde{} = D, \\
&D \mapsto \xi_D.
\end{aligned}$$

□

Proposition 3.42. *Suppose $p \geq q \geq 2$ with $p+q=n$. Let (σ, V) be a representation of P_{n-2}, and let χ be a positive character of $P_{n-1} \cap H^{(n)}_{p,q-1}$. Then there exists a positive character χ' of $P_{n-2} \cap H^{(n)}_{p-1,q-1}$, such that*

$$\operatorname{Hom}_{P_{n-1}\cap H^{(n)}_{p,q-1}}\left(\Phi^+(\sigma), \chi\right) \hookrightarrow \operatorname{Hom}_{P_{n-2}\cap H^{(n)}_{p-1,q-1}}\left(\sigma, \chi'\right).$$

The proof is similar to the proof of the previous proposition. One uses the decomposition $S^{(n)}_{p,q-1} = P_{n-2}H^{(n)}_{p-1,q-1}$ instead of $S^{(n)}_{p,q} = P_{n-1}H^{(n)}_{p,q-1}$.

Now we can prove Theorem 3.37.

Proof. Since π is an irreducible supercuspidal representation, its restriction to P_{2m} equals $\pi \restriction_{P_{2m}} \cong (\Phi^+)^{2m-1}(1)$ ([BZ76, 5.18], [Gel70, Theorem 2.3]). We first show

$$\dim \operatorname{Hom}_{P_{2m}\cap H^{(2m)}_{m,m}}\left(\left(\Phi^+\right)^{2m-1}(1), 1\right) \leq 1.$$

Using Proposition 3.41 and then Proposition 3.42, we obtain the existence of characters $\chi' : P_{2m-1} \cap H^{(2m)}_{m,m-1} \to \mathbb{C}^*$ and $\chi'' : P_{2m-2} \cap H^{(2m)}_{m-1,m-1} \to \mathbb{C}^*$ and embeddings of the following form

$$\begin{aligned}
\operatorname{Hom}_{P_{2m}\cap H^{(2m)}_{m,m}}\left(\left(\Phi^+\right)^{2m-1}(1), 1\right) &\hookrightarrow \operatorname{Hom}_{P_{2m-1}\cap H^{(2m)}_{m,m-1}}\left(\left(\Phi^+\right)^{2m-2}(1), \chi'\right) \\
&\hookrightarrow \operatorname{Hom}_{P_{2m-2}\cap H^{(2m)}_{m-1,m-1}}\left(\left(\Phi^+\right)^{2m-3}(1), \chi''\right).
\end{aligned}$$

Note that the standard embedding of $H^{(2m-2)}_{m-1,m-1}$ in G_{2m} is $H^{(2m)}_{m-1,m-1}$, and therefore

$$\operatorname{Hom}_{P_{2m-2}\cap H^{(2m)}_{m-1,m-1}}\left(\left(\Phi^+\right)^{2m-3}(1), \chi''\right) = \operatorname{Hom}_{P_{2m-2}\cap H^{(2m-2)}_{m-1,m-1}}\left(\left(\Phi^+\right)^{2m-3}(1), \chi''\right).$$

Continuing using Proposition 3.41 and Proposition 3.42 repeatedly, we obtain an embedding

$$\operatorname{Hom}_{P_{2m}\cap H^{(2m)}_{m,m}}\left(\left(\Phi^+\right)^{2m-1}(1), 1\right) \hookrightarrow \operatorname{Hom}_{P_1\cap H^{(2)}_{1,0}}(1,1).$$

Since $P_1 \cap H^{(2)}_{1,0} = \{I_2\}$, we have $\operatorname{Hom}_{P_1\cap H^{(2)}_{1,0}}(1,1) = \mathbb{C}$, and therefore

$$\dim_{\mathbb{C}} \operatorname{Hom}_{P_{2m}\cap H^{(2m)}_{m,m}}\left(\left(\Phi^+\right)^{2m-1}(1), 1\right) \leq \dim_{\mathbb{C}} \mathbb{C} = 1.$$

We now show that $\dim \operatorname{Hom}_{P_{2m}\cap M_{m,m}}(\pi, 1) \leq 1$. We have that

$$w_{m,m} M_{m,m} w_{m,m}^{-1} \cap P_{2m} = w_{m,m}\left(M_{m,m} \cap w_{m,m}^{-1} P_{2m} w_{m,m}\right) w_{m,m}^{-1}.$$

Since $\sigma(2m) = 2m$, we have that $\left(w_{m,m}^{-1} p w_{m,m}\right)_{m,j} = p_{\sigma(m),\sigma(j)} = p_{m,\sigma(j)} = \begin{cases} 1 & j = m \\ 0 & j \neq m \end{cases}$, for $p \in P_{2m}$, and therefore $w_{m,m}^{-1} P_{2m} w_{m,m} \subseteq P_{2m}$. Similarly, since $\sigma^{-1}(m) = m$, we have $w_{m,m} P_{2m} w_{m,m}^{-1} \subseteq P_{2m}$, and therefore $w_{m,m} P_{2m} w_{m,m}^{-1} = P_{2m}$, and we get

$$P_{2m} \cap H_{m,m}^{(2m)} = w_{m,m} \left(P_{2m} \cap M_{m,m}\right) w_{m,m}^{-1}.$$

Therefore we have

$$\operatorname{Hom}_{P_{2m} \cap H_{m,m}^{(2m)}} (\pi, 1) \cong \operatorname{Hom}_{P_{2m} \cap M_{m,m}} (\pi, 1),$$

by mapping $L \in \operatorname{Hom}_{P_{2m} \cap H_{m,m}^{(2m)}} (\pi, 1)$ to $L\pi(w_{m,m})$. Therefore, we get the result

$$\dim_{\mathbb{C}} \operatorname{Hom}_{P_{2m} \cap M_{m,m}} (\pi, 1) \leq 1.$$

□

3.5.2. *An embedding of two homomorphism spaces.* In this subsection, we construct an embedding $\operatorname{Hom}_{P_{2m} \cap S_{2m}} (\pi, \Psi) \hookrightarrow \operatorname{Hom}_{P_{2m} \cap M_{m,m}} (\pi, 1)$. We follow [Mat14, Section 4].

We begin with the following lemma.

Lemma 3.43. *Let π be a representation of $P_{m,m}$, $L \in \operatorname{Hom}_{N_{m,m}} \left(\pi \restriction_{N_{m,m}}, \Psi\right)$ and $v \in V_\pi$. Denote by $S : P_{m,m} \to \mathbb{C}$ the map*

$$S(p) = L(\pi(p) v),$$

and by $\tilde{S} : G_m \to \mathbb{C}$ the map $\tilde{S}(g) = S\left(\left(\begin{smallmatrix} g & \\ & I_m \end{smallmatrix}\right)\right)$. Then there exists a function $\xi \in \mathcal{S}(M_m(F))$, such that for every $g \in G_m$, one has $\tilde{S}(g) = \tilde{S}(g) \xi(g)$. In particular, the integral

$$c_k(S) = \int_{\substack{g \in G_m \\ |\det g| = q^{-k}}} \tilde{S}(g)\, dg$$

converges absolutely for all $k \in \mathbb{Z}$. Moreover $c_k(S) = 0$ for $k \ll 0$.

Proof. Since π is smooth, $\left(\operatorname{stab}_{P_{m,m}} v\right) \cap N_{m,m} \subseteq N_{m,m}$ is an open subgroup of $N_{m,m}$ and contains a compact subgroup. Since the projection homomorphism $N_{m,m} \to M_m(F)$ (defined by $\left(\begin{smallmatrix} I_m & X \\ & I_m \end{smallmatrix}\right) \mapsto X$) is an open map, we get that that there exists an open compact subgroup C of $M_m(F)$ such that $\left(\begin{smallmatrix} I_m & C \\ & I_m \end{smallmatrix}\right)$ stabilizes v.

Let $f : M_m(F) \to \mathbb{C}$ be the indicator function of C, $f = 1\chi_C$. For a Haar measure on $M_m(F)$, normalized by C, we have for every $p \in P_{m,m}$:

$$S(p) = \int_{M_m(F)} S\left(p \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\right) f(X)\, dX.$$

Taking $p = \left(\begin{smallmatrix} g & \\ & I_m \end{smallmatrix}\right)$ and using the fact that L is a homomorphism we get

$$\tilde{S}(g) = \tilde{S}(g) \cdot \int_{M_m(F)} f(X)\, \psi(\operatorname{tr}(gX))\, dX.$$

We denote $\xi(g) = \int_{M_m(F)} f(X)\, \psi(\operatorname{tr}(gX))\, dX$. $\xi(g)$ is the Fourier transform of the function $f \in \mathcal{S}(M_m(X))$, and therefore $\xi \in \mathcal{S}(M_m(X))$.

Since ξ is a Schwartz function, it has compact support. Therefore, if $X \in \operatorname{supp}\xi$, $|\det X|$ is bounded as a continuous image of a compact set, and thus if $|\det X|$ is large, then $\xi(X) = 0$.

Hence, $\tilde{S}(g) = \xi(g)\tilde{S}(g)$ vanishes for g with large $|\det g|$, and therefore $\int_{\substack{g\in G_m \\ |\det g| = q^{-k}}} \tilde{S}(g)\,dg$ vanishes for $k < 0$ from some place.

Finally, for $k \in \mathbb{Z}$ the set $\left\{X \in M_m(F) \mid |\det X| = q^{-k}\right\} = \left\{g \in G_m \mid |\det g| = q^{-k}\right\}$ is closed, and therefore its intersection with suppξ is compact. Since $\tilde{S}(g) = \tilde{S}(g)\xi(g)$, the integral $\int_{\substack{g\in G_m \\ |\det g| = q^{-k}}} \tilde{S}(g)\,dg$ is actually integrated on a compact subset of $M_m(F)$, and therefore converges absolutely. ☐

Lemma 3.44. *Let $S \in \mathrm{Ind}_{N_{m,m}}^{G_{2m}}(\Psi)$. Then there exists $\phi \in \mathcal{S}(M_m \times G_m \times \mathrm{GL}_{2m}(\mathcal{O}))$, such that*

$$S(g) = \int_{M_m} dY \int_{G_m} db \int_{\mathrm{GL}_{2m}(\mathcal{O})} dk S\left(g\begin{pmatrix} b^{-1} & \\ & I_m\end{pmatrix}\begin{pmatrix} I_m & Y \\ & I_m\end{pmatrix}\begin{pmatrix} I_m & \\ & b\end{pmatrix}k\right)\phi(Y,b,k)\,|\det b|^m .$$

Proof. Since S is in $\mathrm{Ind}_{N_{m,m}}^{G_{2m}}(\Psi)$, there exists an open subset $K \subseteq G_{2m}$ such that $S(gk_0) = S(g)$, for every $k_0 \in K$.

The map $M_m \times G_m \times \mathrm{GL}_{2m}(\mathcal{O}) \to G_{2m}$ defined by

$$(Y,b,k) \mapsto \begin{pmatrix} b^{-1} & \\ & I_m\end{pmatrix}\begin{pmatrix} I_m & Y \\ & I_m\end{pmatrix}\begin{pmatrix} I_m & \\ & b\end{pmatrix}k$$

is continuous, and therefore there exists an open subset $C \subseteq M_m \times G_m \times \mathrm{GL}_{2m}(\mathcal{O})$, such that the image of C under this map is contained in K. $M_m \times G_m \times \mathrm{GL}_{2m}(\mathcal{O})$ is an l-group as a product of such, and therefore we may assume that C is compact. The function $\phi(Y,b,k) = \mu(C)^{-1}\chi_C(Y,b,v)\cdot|\det b|^{-m}$ is as requested (Here μ is the Haar measure on $M_m \times G_m \times \mathrm{GL}_{2m}(\mathcal{O})$ given by $\mu(A) = \int_{M_m} dY \int_{G_m} db \int_{\mathrm{GL}_{2m}(\mathcal{O})} dk 1\chi_A(Y,b,k)$). ☐

We now introduce a quite long list of notations. Let π be an irreducible representation of G_{2m} and let $L \in \mathrm{Hom}_{N_{m,m}}(\pi,\Psi)$. For $v \in V_\pi$ we denote $L_v \in \mathrm{Ind}_{N_{m,m}}^{G}(\Psi)$ by $L_v(g) = L(\pi(g)v)$ (Frobenius reciprocity). Let $S = L_v$ for some $v \in V_\pi$. The previous lemma associates (not uniquely) to S a smooth map with compact support $\phi \in \mathcal{S}(M_m \times G_m \times \mathrm{GL}_{2m}(\mathcal{O}))$.

Let C_b be the compact support of $\phi(Y,b,k)$ in the variable $b \in G_m$ and denote by $\phi' : M_m \to \mathbb{C}$ the characteristic function of C_b^{-1}: $\phi'(x) = 1\chi_{C_b^{-1}}(x)$ (Note that since G_m is open in M_m, and C_b^{-1} is open in G_m and compact, we have that C_b^{-1} is an open compact subset of M_m). We denote by Φ the map in the variables $A, X \in M_m$, $b \in G_m$ and $k \in \mathrm{GL}_{2m}(\mathcal{O})$ defined by

$$\Phi\left(\begin{pmatrix} A & X \\ & b\end{pmatrix}, k\right) = \int_{M_m} dY \int_{M_m} dZ \phi(Y,b,k)\,\phi'(Z)\,\psi(\mathrm{tr}(YA - ZX)) .$$

This integral converges absolutely, as the integrand is a smooth function with compact support in both variables Y, Z.

Φ can be written as a product of Fourier transforms of two Schwartz functions:

$$\Phi\left(\begin{pmatrix} A & X \\ & b\end{pmatrix}, k\right) = \int_{M_m} \phi(Y,b,k)\,\psi(\mathrm{tr}(YA))\,dY \cdot \int_{M_m} \phi'(Z)\,\psi(\mathrm{tr}(-ZX))\,dZ.$$

It follows at once that as such, Φ is smooth and has compact support in the variables $(A,X,b,k) \in M_m \times M_m \times G_m \times \mathrm{GL}_{2m}(\mathcal{O})$.

Lemma 3.45. *For S, ϕ and Φ as above and $a, b \in G_m$, the integrals*

$$I(S,\Phi,a,b) = \int_{M_m} dX \int_{\mathrm{GL}_{2m}(\mathcal{O})} dk S\left(\begin{pmatrix} a & X \\ & b \end{pmatrix} k\right) \Phi\left(\begin{pmatrix} a & X \\ & b \end{pmatrix}, k\right),$$

$$J(S,\phi,a,b) = \int_{M_m} dY \int_{\mathrm{GL}_{2m}(\mathcal{O})} dk S\left(\begin{pmatrix} a & \\ & I_m \end{pmatrix}\begin{pmatrix} I_m & Y \\ & I_m \end{pmatrix}\begin{pmatrix} I_m & \\ & b \end{pmatrix} k\right) \phi(Y,b,k),$$

both converge absolutely, and are equal. They define a map which is smooth with respect to the variables $a \in G_m$, $b \in G_m$. The map's support is contained in a compact subset of $M_m \times G_m$.

Proof. Since the maps $X \mapsto \Phi\left(\left(\begin{smallmatrix} a & X \\ & b \end{smallmatrix}\right), k\right)$ and $Y \mapsto \phi(Y,b,k)$ have compact support in the variables X and Y respectively, the integrals are actually integrated on compact sets. These integrals converge absolutely, as their corresponding integrands are smooth functions on compact sets.

We define $f : G_m \times M_m \times G_m \mathrm{GL}_{2m}(\mathcal{O}) \to \mathbb{C}$ by $f(a,X,b,k) = S\left(\begin{pmatrix} a & X \\ & b \end{pmatrix} k\right) \Phi\left(\begin{pmatrix} a & X \\ & b \end{pmatrix}, k\right)$. Since $S \in \mathrm{Ind}_{N_{m,m}}^{G_{2m}}(\Psi)$, we have

$$f(a,X,b,k) = \psi\left(\mathrm{tr}\left(Xb^{-1}\right)\right) S\left(\begin{pmatrix} a & \\ & b \end{pmatrix} k\right) \Phi\left(\begin{pmatrix} a & X \\ & b \end{pmatrix}, k\right).$$

By substituting the definition of Φ we get

$$\begin{aligned}\int_{M_m} f(a,X,b,k)\, dX =& S\left(\begin{pmatrix} a & \\ & b \end{pmatrix} k\right) \int_{M_m} \phi(Y,b,k)\, \psi(\mathrm{tr}(Ya))\, dY \cdot \\ & \cdot \int_{M_m} \psi\left(\mathrm{tr}\left(Xb^{-1}\right)\right) \left(\int_{M_m} \phi'(Z)\, \psi(\mathrm{tr}(-ZX))\, dZ\right) dX.\end{aligned}$$

We notice that the integral $\int_{M_m} \phi'(Z)\, \psi(\mathrm{tr}(-ZX))\, dZ$ is the Fourier transform of ϕ' at the point $-X$, and therefore

$$\int_{M_m} \psi\left(\mathrm{tr}\left(Xb^{-1}\right)\right) \left(\int_{M_m} \phi'(Z)\, \psi(\mathrm{tr}(-ZX))\, dZ\right) dX = \int_{M_m} \psi\left(\mathrm{tr}\left(-X'b^{-1}\right)\right) \widehat{\phi'}(X')\, dX',$$

which equals the value of the Fourier transform of $\widehat{\phi'}$ at the point $-b^{-1}$. By Fourier's inversion formula we get that

$$\int_{M_m} f(a,X,b,k)\, dX = \int_{M_m} S\left(\begin{pmatrix} a & \\ & b \end{pmatrix} k\right) \phi'\left(b^{-1}\right) \phi(Y,b,k)\, \psi(\mathrm{tr}(Ya))\, dY.$$

Since ϕ' is the indicator function of C_b^{-1}, where C_b is the support of ϕ in the variable b, we have that $\phi(Y,b,k)$ vanishes whenever $\phi'(b^{-1})$ vanishes, and therefore

$$\int_{M_m} f(a,X,b,k)\, dX = \int_{M_m} \phi(Y,b,k)\, S\left(\begin{pmatrix} a & \\ & b \end{pmatrix} k\right) \psi(\mathrm{tr}(Ya))\, dY.$$

Finally, using again the fact that $S \in \mathrm{Ind}_{N_{m,m}}^{G_{2m}}(\Psi)$ we have

$$\psi(\mathrm{tr}(Ya))\, S\left(\begin{pmatrix} a & \\ & b \end{pmatrix} k\right) = S\left(\begin{pmatrix} a & \\ & I_m \end{pmatrix}\begin{pmatrix} I_m & Y \\ & I_m \end{pmatrix}\begin{pmatrix} I_m & \\ & b \end{pmatrix} k\right),$$

and we get

$$\int_{M_m} f(a,X,b,k)\,dX = \int_{M_m} \phi(Y,b,k)\, S\left(\begin{pmatrix} a & \\ & I_m \end{pmatrix}\begin{pmatrix} I_m & Y \\ & I_m \end{pmatrix}\begin{pmatrix} I_m & \\ & b \end{pmatrix} k\right) dY.$$

Integrating both expressions for $\int_{M_m} f(a,X,b,k)\,dX$ by k on $\mathrm{GL}_{2m}(\mathcal{O})$, yields the desired equality.

We now move to explain why the integrals define a smooth function whose support is contained in a compact subset of $M_m \times G_m$. Using Proposition 3.8 with the compact set $\mathrm{supp}\phi$ and $G = G_{2m}$, the map $(Y,b,k) \mapsto \left(\begin{smallmatrix} I_m & Y \\ & I_m \end{smallmatrix}\right)\left(\begin{smallmatrix} I_m & \\ & b \end{smallmatrix}\right) k$, the representation $\mathrm{Ind}_{N_{m,m}}^{G_{2m}}(\pi)$ and the vector $v = S$, we get that there exists a sequence $(S_i)_{i=1}^N \subseteq \mathrm{Ind}_{N_{m,m}}^{G_{2m}}(\pi)$ and a sequence $(\alpha_i)_{i=1}^N$ of smooth functions $\alpha_i : \mathrm{supp}\phi \to \mathbb{C}$, such that

$$\rho\left(\begin{pmatrix} I_m & Y \\ & I_m \end{pmatrix}\begin{pmatrix} I_m & \\ & b \end{pmatrix} k\right) S = \sum_{i=1}^N \alpha_i(Y,b,k)\, S_i,$$

for every $(Y,b,k) \in \mathrm{supp}\phi$. We extend the definition of α_i to the set $M_m \times G_m \times \mathrm{GL}_{2m}(\mathcal{O})$ by defining it to be zero outside of $\mathrm{supp}\phi$. This is still a smooth function, as $\mathrm{supp}\phi$ is closed in the larger set.

We have that

$$S\left(\begin{pmatrix} a & \\ & I_m \end{pmatrix}\begin{pmatrix} I_m & Y \\ & I_m \end{pmatrix}\begin{pmatrix} I_m & \\ & b \end{pmatrix} k\right) = \sum_{i=1}^N \alpha_i(Y,b,k)\, \widetilde{S_i}(a),$$

and therefore

$$J(S,\phi,a,b) = \sum_{i=1}^N \widetilde{S_i}(a) \int_{M_m} dY \int_{\mathrm{GL}_{2m}(\mathcal{O})} dk\,\alpha_i(Y,b,k)\,\phi(Y,b,k).$$

$\widetilde{S_i}$ is smooth, since $\mathrm{Ind}_{N_{m,m}}^{G_{2m}}(\Psi)$ is smooth. $\alpha_i \cdot \phi$ is smooth as well in the variable b, and therefore the integral defines a smooth function.

Finally, $\widetilde{S_i} = \widetilde{S_i} \cdot \xi_i$, where $\xi_i \in \mathcal{S}(M_m)$, and therefore $\mathrm{supp}\widetilde{S_i} \subseteq \mathrm{supp}\xi_i$, where $\mathrm{supp}\xi_i$ is compact. We get immediately that the support of the function that this integral defines is contained in $\bigcup_{i=1}^N (\mathrm{supp}\xi_i) \times (\mathrm{supp}_b\phi)$. This finite union is a compact subset of $M_m \times G_m$. ☐

Let

$$\Omega = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A,B,C,D \in M_m \mid \begin{pmatrix} C & D \end{pmatrix} \text{ has rank } m \right\}.$$

Then Ω is an open subset of M_{2m} as having rank m is equivalent for having a non-zero minor of order m. We denote

$$\Omega_0 = \left\{ \begin{pmatrix} A & B \\ & d \end{pmatrix} \mid A,B \in M_m,\ d \in G_m \right\}.$$

Using the same elimination algorithm used in the proof of the Iwasawa decomposition, one gets that the multiplication map $r : \Omega_0 \times \mathrm{GL}_{2m}(\mathcal{O}) \to \Omega$, $r(p,k) = pk$ is surjective.

We define a map $\Phi_* : \Omega \to \mathbb{C}$ by

$$\Phi_*(pk) = \int_{k' \in \mathrm{GL}_{2m}(\mathcal{O}) \cap P_{m,m}} \Phi\left(pk'^{-1}, k'k\right) dk',$$

for $p \in \Omega_0$, $k \in \mathrm{GL}_{2m}(\mathcal{O})$. This map is well defined: if $p_1 k_1 = p_2 k_2$ then $p_1 = p_2 k_2 k_1^{-1}$. Writing $p_1 = \left(\begin{smallmatrix} A & B \\ 0 & d \end{smallmatrix}\right)$, $p_2 = \left(\begin{smallmatrix} A' & B' \\ 0 & d' \end{smallmatrix}\right)$, $k_2 k_1^{-1} = \left(\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix}\right)$ implies $d' \cdot C'' = 0$, and since d' is invertible, this implies $C'' = 0$, and therefore $k_2 k_1^{-1} \in P_{m,m} \cap \mathrm{GL}_{2m}(\mathcal{O})$. Translating the integral in the definition of Φ_* by $k_2 k_1^{-1}$ from the right, we get

$$\int_{k' \in \mathrm{GL}_{2m}(\mathcal{O}) \cap P_{m,m}} \Phi\left(p_1 k'^{-1}, k' k_1\right) dk' = \int_{k'' \in \mathrm{GL}_{2m}(\mathcal{O}) \cap P_{m,m}} \Phi\left(p_2 k''^{-1}, k'' k_2\right) dk''.$$

Since Φ is smooth with compact support, there exists an open compact subgroup of $\mathrm{GL}_{2m}(\mathcal{O})$, such that Φ is invariant to right multiplication of the variable k under this subgroup. Therefore the map Φ_* is fixed by right multiplication under a compact open subgroup of $\mathrm{GL}_{2m}(\mathcal{O})$. Similarly, there exist open compact subgroups $C_A \subseteq M_m$, $C_X \subseteq M_m$, $C_d \subseteq G_m$, such that, for any $\left(\begin{smallmatrix} A_0 & X_0 \\ & d_0 \end{smallmatrix}\right) \in \Omega_0$, $A' \in C_A$, $X' \in C_X$, $d' \in C_d$ and $k \in \mathrm{GL}_{2m}(\mathcal{O})$

$$\Phi\left(\begin{pmatrix} A' + A_0 & X' + X_0 \\ & d' d_0 \end{pmatrix}, k\right) = \Phi\left(\begin{pmatrix} A_0 & X_0 \\ & d_0 \end{pmatrix}, k\right).$$

Choosing the subgroups such that $C_A = C_X \subseteq M_m(\mathcal{O})$ implies that $\Phi_*\left(\left(\begin{smallmatrix} A'+A_0 & X'+X_0 \\ & d'd_0 \end{smallmatrix}\right) k\right) = \Phi_*\left(\left(\begin{smallmatrix} A_0 & X_0 \\ & d_0 \end{smallmatrix}\right) k\right)$. Combining these facts, we get that Φ_* is smooth.

It follows that $\mathrm{supp}\Phi_*$ is closed. It is clear that $\mathrm{supp}\Phi_* \subseteq r(\mathrm{supp}\Phi) \cdot \mathrm{GL}_{2m}(\mathcal{O})$, where r is again the multiplication map. Since $\mathrm{supp}\Phi$ and $\mathrm{GL}_{2m}(\mathcal{O})$ are compact, we get that $r(\mathrm{supp}\Phi) \cdot \mathrm{GL}_{2m}(\mathcal{O})$ is compact, and therefore $\mathrm{supp}\Phi_*$ is compact, as a closed subset of this compact set.

We wish to extend the definition of Φ_* to a Schwartz function on M_{2m}, in order to be able to use it for a Godement-Jacquet integral (Theorem 3.4) in the proof of Proposition 3.47. Note that $\mathrm{supp}\Phi_*$ is open and compact. Therefore, we can extend $\Phi_* : M_{2m} \to \mathbb{C}$ to a Schwartz function on M_{2m}, by defining Φ_* as zero outside of Ω.

Let U be a compact open subgroup of $\mathrm{GL}_{2m}(\mathcal{O})$, such that Φ_* is invariant under left multiplication by U. We define for $S(g) = L(\pi(g)v)$ where $v \in V_\pi$,

$$S^U(g) = \int_U S\left(u^{-1} g\right) du,$$

where du is a normalized Haar measure of U. S^U is a matrix coefficient of π: the functional $\tilde{L} : V_\pi \to \mathbb{C}$ defined by $\tilde{L}(v) = \int_U L\left(\pi\left(u^{-1}\right) v\right) du$ is smooth, since it is invariant to the action of U, and therefore $S^U(g) = \left\langle \tilde{L}, \pi(g) v \right\rangle$ is indeed a matrix coefficient.

For $k, l \in \mathbb{Z}$ we define

$$a_{k,l}(\Phi, S) = q^{-lm} \int_{\substack{|\det a| = q^{-k} \\ |\det b| = q^{-l}}} I(S, \Phi, a, b)\, da db,$$

$$b_{k,l}(\phi, S) = q^{-lm} \int_{\substack{|\det a| = q^{-k} \\ |\det b| = q^{-l}}} J(S, \phi, a, b)\, da db.$$

Note that

$$\left\{(a,b) \in G_m \times G_m \mid |\det a| = q^{-k}, |\det a| = q^{-l}\right\} = \left\{(a,b) \in M_m \times G_m \mid |\det a| = q^{-k}, |\det a| = q^{-l}\right\},$$

is a closed subset of $M_m \times G_m$. Since the support of $I(S,\Phi,a,b) = J(S,\phi,a,b)$ (with respect to the variables a,b) is contained in a compact subset of $M_m \times G_m$, this integral is actually integrated on a compact set (as an intersection of a closed set and a compact set), and since the integrand is smooth, the integral converges absolutely.

Furthermore, since the support of $I(S,\Phi,a,b) = J(S,\Phi,a,b)$ (with respect of the variables a,b) is contained in a compact subset of $M_m \times G_m$, the image of map $(a,b) \mapsto (|\det a|, |\det b|)$ is bounded for a,b in the support, and therefore $I(S,\Phi,a,b)$ vanishes for $a,b \in G_m$ with large determinant. This implies that $a_{k,l}(\Phi,S) = 0$ for $k,l \ll 0$. Moreover, since $b \in G_m$, $|\det b|$ is also bounded from below, i.e. $a_{k,l}(\Phi,S) = 0$ for $l \gg 0$.

We now define

$$I(S,\Phi_*,a,b) = \int_{M_m} dX \int_{\mathrm{GL}_{2m}(\mathcal{O})} dk_0 S\left(\begin{pmatrix} a & X \\ & b \end{pmatrix} k_0\right) \Phi_*\left(\begin{pmatrix} a & x \\ & b \end{pmatrix} k_0\right),$$

$$a_{k,l}(S,\Phi_*) = q^{-lm} \int_{\substack{|\det a| = q^{-k} \\ |\det b| = q^{-l}}} I(S,\Phi_*,a,b)\, da db.$$

Claim 3.46. $a_{k,l}(S,\Phi) = a_{k,l}(S,\Phi_*)$.

Proof. One substitutes the definitions of $I(S,\Phi_*,a,b)$ and Φ_* to the expression

$$\int_{\substack{|\det a| = q^{-k} \\ |\det b| = q^{-l}}} I(S,\Phi_*,a,b)\, da db.$$

Note that for $k' \in \mathrm{GL}_{2m}(\mathcal{O}) \cap P_{m,m}$ and $\left(\begin{smallmatrix} a & X \\ & b \end{smallmatrix}\right) \in P_{m,m}$, one has $\left(\begin{smallmatrix} a & X \\ & b \end{smallmatrix}\right) = \left(\begin{smallmatrix} a' & X' \\ & b' \end{smallmatrix}\right) k'$, where $\left(\begin{smallmatrix} a' & X' \\ & b' \end{smallmatrix}\right) \in P_{m,m}$. Substituting $\left(\begin{smallmatrix} a & X \\ & b \end{smallmatrix}\right) = \left(\begin{smallmatrix} a' & X' \\ & b' \end{smallmatrix}\right) k'$ (in the same notations as in the definitions), and then substituting $k_0 = k'^{-1} k''$ (for the integration with respect to $k_0 \in \mathrm{GL}_{2m}(\mathcal{O})$) yields the desired equality. □

Proposition 3.47. *The sum* $\sum_{j \in \mathbb{Z}} \left| \sum_{\substack{k,l \in \mathbb{Z} \\ k+l=j}} a_{k,l}(S,\Phi) q^{-ks} q^{-ls} \right|$ *converges for* $\mathrm{Re}(s)$ *greater than a real* r_π *depending only on* π. *In particular the sum*

$$\sum_{j \in \mathbb{Z}} \sum_{\substack{k,l \in \mathbb{Z} \\ k+l=j}} a_{k,l}(S,\Phi) q^{-ks} q^{-ls}$$

converges for $\mathrm{Re}(s) > r_\pi$ *for the same* r_π. *The latter sum extends meromorphically to an element of* $L\left(\pi, s+\frac{1}{2}\right) \mathbb{C}[q^s, q^{-s}]$.

Proof. As seen before, $a_{k,l}(S,\Phi) = a_{k,l}(S,\Phi_*)$. For a fixed j the sum

$$d_j(S,\Phi) = \sum_{\substack{k,l \in \mathbb{Z} \\ k+l=j}} a_{k,l}(S,\Phi_*) q^{-ks} q^{-ls}$$

is a finite sum, as we have seen that $a_{k,l}(S,\Phi)$ vanishes for $k,l \ll 0$. A simple calculation shows that

$$d_j(S,\Phi) = q^{-j(m+s)} \int_{\left|\det\left(\left(\begin{smallmatrix} a & X \\ & b \end{smallmatrix}\right)k_0\right)\right| = q^{-j}} dadb \int_{M_m} dX \int_{\mathrm{GL}_{2m}(\mathcal{O})} dk_0$$
$$\frac{1}{|\det a|^m} S\left(\begin{pmatrix} a & X \\ & b \end{pmatrix} k_0\right) \Phi_*\left(\begin{pmatrix} a & X \\ & b \end{pmatrix} k_0\right).$$

To proceed, we use the following expression for the Haar measure on G_{2m}:

$$\int_{G_{2m}} f(g)\, dg = \int_{G_m} da \int_{G_m} db \int_{M_m} dX \int_{\mathrm{GL}_{2m}(\mathcal{O})} dk_0 \frac{1}{|\det a|^m} f\left(\begin{pmatrix} a & X \\ & b \end{pmatrix} k_0\right).$$

Therefore

$$d_j(S,\Phi) = \int_{|\det g| = q^{-j}} S(g)\, \Phi_*(g)\, |\det g|^{m+s}\, dg.$$

Since Φ_* is invariant under left translations of U, we have

$$d_j(S,\Phi) = \int_{|\det g| = q^{-j}} S^U(g)\, \Phi_*(g)\, |\det g|^{m+s}\, dg,$$

hence

$$|d_j(S,\Phi)| \le \int_{|\det g| = q^{-j}} \left|S^U(g)\right| |\Phi_*(g)|\, |\det g|^{m+\mathrm{Re}(s)}\, dg.$$

Summing on j yields

$$\sum_{j\in\mathbb{Z}} \left| \sum_{\substack{k,l \\ k+l=j}} a_{k,l}(S,\Phi)\, q^{-ks} q^{-ls} \right| \le \int_{G_{2m}} \left|S^U(g)\right| |\Phi_*(g)|\, |\det g|^{m+\mathrm{Re}(s)}\, dg.$$

The integral $\int_G S^U(g)\,\Phi_*(g)\cdot|\det g|^{m+s}\,dg$ is a local zeta integral of Godement and Jacquet, and therefore by Theorem 3.4, it converges absolutely for $\mathrm{Re}(s) > r_\pi$, where r_π is a real number depending on π only, to an element of $L\left(\pi, s+\frac{1}{2}\right)\mathbb{C}[q^s, q^{-s}]$. Finally, since the series converges for $\mathrm{Re}(s) > r_\pi$, we get

$$\sum_{j\in\mathbb{Z}} \sum_{\substack{k,l\in\mathbb{Z} \\ k+l=j}} a_{k,l}(S,\Phi)\, q^{-ks} q^{-ls} = \int_{G_{2m}} S^U(g)\, \Phi_*(g)\, |\det g|^{m+s}\, dg,$$

and therefore the sum $\sum_{j\in\mathbb{Z}} \sum_{\substack{k,l\in\mathbb{Z} \\ k+l=j}} a_{k,l}(S,\Phi)\, q^{-ks} q^{-ls}$ has a meromorphic continuation to an element of $L\left(\pi, s+\frac{1}{2}\right)\mathbb{C}[q^s, q^{-s}]$. □

Proposition 3.48. *The sum $I(S,s) = \sum_{k\in\mathbb{Z}} c_k(S)\, q^{-ks}$ converges absolutely for $\mathrm{Re}(s) > r_\pi$. It equals to the sum $\sum_{j\in\mathbb{Z}} \sum_{\substack{k,l\in\mathbb{Z} \\ k+l=j}} a_{k,l}(S,\Phi)\, q^{-ks} q^{-ls}$.*

Proof. We write for a fixed $l \in \mathbb{Z}$, $\sum_{k\in\mathbb{Z}} b_{k,l}(S,\phi)\, q^{-ks} = \sum_{k\in\mathbb{Z}} b_{k-l,l}(S,\phi)\, q^{-(k-l)s}$. We have seen that $b_{k,l}(S,\phi) = 0$, for $l \gg 0$ and $l \ll 0$ uniformly, with respect to k. A simple calculation shows

$$\sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} b_{k,l}(S,\phi)\,q^{-ks}q^{-ls}=\sum_{k\in\mathbb{Z}}\left(\sum_{l\in\mathbb{Z}} b_{k-l,l}(S,\phi)\right)q^{-ks}.$$

Substituting the definitions of $b_{k-l,l}(S,\phi)$ and $J(S,\phi,a,b)$ and substituting (in the notations of the definitions) $a=a'b^{-1}$, $|\det a'|=q^{-k}$, we get that the sum $\sum_{l\in\mathbb{Z}} b_{k-l,l}(S,\phi)$ equals

$$\int_{G_m} db\int_{|\det a'|=q^{-k}} da'\int_{M_m} dY\int_{\mathrm{GL}_{2m}(\mathcal{O})} dk_0 S\left(\begin{pmatrix}a'b^{-1} & \\ & I_m\end{pmatrix}\begin{pmatrix}I_m & Y\\ & I_m\end{pmatrix}\begin{pmatrix}I_m & \\ & b\end{pmatrix}k_0\right)\phi(Y,b,k_0)\,|\det b|^m.$$

Recalling that ϕ was chosen by Lemma 3.44, we get that $\sum_{l\in\mathbb{Z}} b_{k-l,l}(S,\phi)=c_k(S)$ (See Lemma 3.43).

Since $a_{k,l}(S,\Phi)=b_{k,l}(S,\phi)$, we have

$$c_k(S)\,q^{-ks}=\sum_{l\in\mathbb{Z}} b_{k-l,l}(S,\phi)\,q^{-ks}=\sum_{\substack{l,l'\in\mathbb{Z}\\ l+l'=k}} a_{l',l}(S,\phi)\,q^{-ls}q^{-l's}.$$

The proposition now follows from Proposition 3.47. □

Corollary 3.49. *The series $I(S,s)=\sum_{k\in\mathbb{Z}} c_k(S)\,q^{-ks}$ has a meromorphic continuation to an element of $L\left(\pi,s+\frac{1}{2}\right)\mathbb{C}[q^s,q^{-s}]$, which we continue to denote $I(S,s)$.*

Proposition 3.50. *Let π be an irreducible representation of G_{2m}. Let $L\in\mathrm{Hom}_{P_{2m}\cap S_{2m}}(\pi,\Psi)$, $v\in V_\pi$, $s\in\mathbb{C}$. Then for $L_v(g)=L(\pi(g)v)$ and $p\in P_{2m}\cap M_{m,m}$ one has*

$$I\left(L_{\pi(p)v},s\right)=\chi(p)^s\cdot I(L_v,s),$$

where $\chi: P_{2m}\cap M_{m,m}\to\mathbb{C}^*$ is defined as $\chi\left(\begin{pmatrix}g_0 & \\ & p_0\end{pmatrix}\right)=\left|\det\left(p_0\cdot g_0^{-1}\right)\right|$, for $p_0\in P_m$, $g_0\in G_m$.

Proof. One writes the definition of $c_k\left(L_{\pi(p)v}\right)$, for $p=\begin{pmatrix}g_0 & \\ & p_0\end{pmatrix}$, where $g_0\in G_m$, $p_0\in P_m$. By conjugating with $\begin{pmatrix}p_0 & \\ & p_0\end{pmatrix}\in S_{2m}\cap P_{2m}$ and substituting $g=p_0g'g_0^{-1}$ $|\det g|=|\det g'|\cdot q^{-k_0}$ where $\left|\det\left(p_0g_0^{-1}\right)\right|=q^{-k_0}$, one gets

$$c_k\left(L_{\pi(p)v}\right)=c_{k-k_0}(L_v).$$

Therefore for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>r_\pi$

$$\sum_{k\in\mathbb{Z}} c_k\left(L_{\pi(p)v}\right)q^{-ks}=q^{-k_0s}\sum_{k\in\mathbb{Z}} c_k(L_v)\,q^{-ks},$$

as requested. By the uniqueness of the meromorphic continuation, this equality remains valid for the meromorphic continuation of $I(L_v,s)$. □

Proposition 3.51. *Let π be an irreducible supercuspidal representation of G_{2m}. The vector space $\mathrm{Hom}_{P_{2m}\cap S_{2m}}(\pi,\Psi)$ embeds as a subspace of $\mathrm{Hom}_{P_{2m}\cap M_{m,m}}(\pi,1)$.*

Proof. As seen in Corollary 3.49, the series $I(S,s)$ extends meromorphically to an element of $L\left(\pi,s+\frac{1}{2}\right)\mathbb{C}[q^s,q^{-s}]$. Since π is supercuspidal, $L(\pi,s)\equiv 1$ (Theorem 3.5), and therefore $I(S,s)$ is defined for every $s\in\mathbb{C}$. Given $L\in\mathrm{Hom}_{P_{2m}\cap S_{2m}}(\pi,\Psi)$ we define $\Lambda(L)$ by

$$\Lambda(L)(v)=I(L_v,0)\qquad\qquad (v\in V_\pi).$$

We have shown that for $p\in P_{2m}\cap M_{m,m}$, we have $I\left(L_{\pi(p)v},s\right)=\chi(p)^s\cdot I(L_v,s)$, and therefore $\Lambda(L)(\pi(p)v)=\Lambda(L)(v)$, i.e. $\Lambda(L)\in\mathrm{Hom}_{P_{2m}\cap M_{m,m}}(\pi,1)$. Λ is a linear map, since it is clear from the definition of $I(S,s)$ that for a fixed $s \in \mathbb{C}$ with $\mathrm{Re}(s) > r_\pi$, we have that $I(\cdot, s)$ is linear.

We claim that Λ is injective. To show that we show that given $L \neq 0$, there exists a vector $v \in V_\pi$, such that $\Lambda(L)(v) \neq 0$.

Let $L \neq 0$ and let $v_0 \in V_\pi$, such that $L(v_0) \neq 0$. By multiplying by a scalar, we may assume $L(v_0) = 1$. Given a Schwartz function $\eta \in \mathcal{S}(M_m)$, we define the vector

$$v_{0,\eta} = \int_{M_m} \eta(X)\, \pi\left(\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\right) v_0 \, dX.$$

(since π is smooth, the integrand is a smooth function of X).

A simple computation shows

$$L\left(\begin{pmatrix} g & \\ & I_m \end{pmatrix} v_{0,\eta}\right) = \underbrace{\int_{M_m} \eta(x)\, \psi(\mathrm{tr}(gX))\, dX}_{\hat{\eta}(g)} \cdot L\left(\pi\left(\begin{pmatrix} g & \\ & I_m \end{pmatrix}\right) v_0\right).$$

Since π is smooth, there exists an open compact subgroup of G_m, which we denote $K_{v_0} \subseteq G_m$, such that $\pi\left(\left(\begin{smallmatrix} k & \\ & I_m \end{smallmatrix}\right)\right) v_0 = v_0$, for every $k \in K_{v_0}$. Since $G_m \subseteq M_m$ is open, $K_{v_0} \subseteq M_m$ is open and compact. Furthermore, we may assume that $K_{v_0} \subseteq \mathrm{GL}_m(\mathcal{O})$. Therefore we have that the indicator function $1\chi_{K_{v_0}} \in \mathcal{S}(M_m)$ is a Schwartz function on M_m. Since the Fourier transform is a bijection from $\mathcal{S}(M_m)$ to itself, there exists $\eta \in \mathcal{S}(M_m)$ such that $\hat{\eta} = \frac{1}{\mu_{G_m}(K_{v_0})} 1\chi_{K_{v_0}}$. Choosing this η yields $L\left(\left(\begin{smallmatrix} g & \\ & I_m \end{smallmatrix}\right) v_{0,\eta}\right) = L(v_0) \cdot 1\chi_{K_{v_0}}(g)$, and therefore for $s > r_\pi$ we have

$$I\left(L_{v_{0,\eta}}, s\right) = \frac{1}{\mu_{G_m}(K_{v_0})} \int_{K_{v_0}} \underbrace{L(v_0)}_{=1} \, dg = 1.$$

Therefore we have shown that for every $L \neq 0$, there exists a vector $v = v_{0,\eta}$, such that $I(L_v, s) \equiv 1$, and therefore the meromorphic continuation $I(L_v, s)$ satisfies $\Lambda(L)(v) = I(L_v, 0) = 1 \neq 0$. ∎

Corollary 3.52. *Let π be an irreducible supercuspidal representation of G_{2m}. Then*

$$\dim \mathrm{Hom}_{P_{2m} \cap S_{2m}}(\pi, \Psi) \leq 1$$

Proof. Combine Theorem 3.37 and Proposition 3.51. ∎

3.5.3. *Proof of the functional equation.* We move to the proof of the functional equation (Theorem 3.36).

Proof. We recall that for a fixed $s \in \mathbb{C}$, the forms $J_{\pi,\psi}, \tilde{J}_{\pi,\psi}$ are $|\det|^{-\frac{s}{2}} \cdot \Psi$ equivariant bilinear maps over S_{2m} and therefore define elements in $\mathrm{Hom}_{S_{2m}}\left(\pi \otimes \mathcal{S}(F^m), |\det|^{-\frac{s}{2}} \cdot \Psi\right)$. We show that the dimension of $\mathrm{Hom}_{S_{2m}}\left(\pi \otimes \mathcal{S}(F^m), |\det|^{-\frac{s}{2}} \cdot \Psi\right)$ is at most 1, for all values of q^{-s}, except for a finite number of values.

We first show that $\mathrm{Hom}_{S_{2m}}\left(\pi \otimes \mathcal{S}(F^m), |\det|^{-\frac{s}{2}} \cdot \Psi\right)$ is embedded as a subspace of $\mathrm{Hom}_{S_{2m}}\left(\pi \otimes \mathcal{S}_0(F^m), |\det|^{-\frac{s}{2}} \cdot \Psi\right)$, for all values of q^{-s}, except for a finite number of values.

Here

$$\mathcal{S}_0(F^m) = \{f \in \mathcal{S}(F^m) \mid f(0) = 0\}.$$

Note that $\mathcal{S}_0(F^m)$ is an invariant subspace of $\mathcal{S}(F^m)$ as the kernel of the homomorphism $f \mapsto f(0)$. We show that the restriction map

$$\text{(3.13)} \qquad \begin{aligned} \mathrm{Hom}_{S_{2m}}\left(\pi \otimes \mathcal{S}(F^m), |\det|^{-\frac{s}{2}} \cdot \Psi\right) &\to \mathrm{Hom}_{S_{2m}}\left(\pi \otimes \mathcal{S}_0(F^m), |\det|^{-\frac{s}{2}} \cdot \Psi\right) \\ b &\mapsto b \restriction_{\pi \otimes \mathcal{S}_0(F^m)}, \end{aligned}$$

is injective. Suppose $b \neq 0$ is a bilinear $|\det|^{-\frac{s}{2}} \cdot \Psi$ equivariant map, such that its restriction to $V_\pi \times \mathcal{S}_0(F^m)$ is the zero map.

We define a bilinear map $\tilde{b} : V_\pi \times {}^{\mathcal{S}(F^m)}/_{\mathcal{S}_0(F^m)} \to \mathbb{C}$ by

$$\tilde{b}(v, f + \mathcal{S}_0(F^m)) = b(v, f).$$

One easily checks that this map is well defined, as b is identically zero on $V_\pi \times \mathcal{S}_0(F^m)$, and that this map is also $|\det|^{-\frac{s}{2}} \cdot \Psi$-equivariant over S_{2m}.

On the other hand, ${}^{\mathcal{S}(F^m)}/_{\mathcal{S}_0(F^m)} \cong \mathbb{C}$ with the trivial representation and therefore we have

$$\tilde{b}(\pi(g)v, \rho(g)(f + \mathcal{S}_0(F^m))) = \tilde{b}(\pi(g)v, f + \mathcal{S}_0(F^m)).$$

Choosing g in the center of G, i.e. $g = \lambda I_n \in S_{2m}$ we have $\pi(\lambda I_n)v = \omega_\pi(\lambda) \cdot v$ where ω_π is the central character of π. Therefore we get

$$\tilde{b}(\pi(\lambda I_{2m})v, \rho(\lambda I_{2m})f + \mathcal{S}_0(F^m)) = |\lambda|^{-ms} \cdot \tilde{b}(v, f + \mathcal{S}_0(F^m)),$$

and on the other hand

$$\tilde{b}(\pi(\lambda I_{2m})v, \rho(\lambda I_{2m})f + \mathcal{S}_0(F^m)) = \omega_\pi(\lambda) \cdot \tilde{b}(v, f + \mathcal{S}_0(F^m)).$$

Choosing values of v, f, such that $b(v, f) \neq 0$, and therefore $\tilde{b}(v, f + \mathcal{S}_0(F^m)) \neq 0$, yields

$$\omega_\pi(\lambda) = |\lambda|^{-ms}.$$

Substituting $\lambda = \varpi$ yields $\omega_\pi(\varpi) = q^{ms}$. Since ω_π depends on π only, this equality can be true only for at most m values of q^{-s} ($q^s = \omega_\pi(\varpi)^{\frac{1}{m}} e^{\frac{2\pi i k}{m}}$, $k \in \{0, \dots, m-1\}$). Therefore, we have shown that except for a finite number of values of q^s, the restriction map defined in (3.13) is injective.

We consider the right action of S_{2m} on row vectors F^m defined by

$$(a_1, \dots, a_m) \cdot \begin{pmatrix} g & x \\ & g \end{pmatrix} = (a_1, \dots, a_m) g.$$

This action has exactly two orbits: $\{0\}$ and $F^m \setminus \{0\}$. The stabilizer of the element $\varepsilon = (0, \dots, 0, 1) \in F^m$ consists of elements of the form $\left(\begin{smallmatrix} g & x \\ & g \end{smallmatrix}\right)$ with $g \in P_m$, i.e.

$$\mathrm{stab}_{S_{2m}}(\varepsilon) = S_{2m} \cap P_{2m}.$$

We have the following homeomorphism ${}_{S_{2m} \cap P_{2m}}\backslash^{S_{2m}} \cong F^m \backslash \{0\}$. Since $\mathcal{S}_0(F^m) \cong \mathcal{S}(F^m \setminus \{0\})$, we get using these identifications that

$$\mathcal{S}_0(F^m) \cong \mathcal{S}\left({}_{S_{2m} \cap P_{2m}}\backslash^{S_{2m}}\right) \cong \mathrm{ind}_{S_{2m} \cap P_{2m}}^{S_{2m}}(1),$$

and therefore we have the following isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{S_{2m}}\left(\pi\otimes\mathcal{S}_0\left(F^m\right),\left|\det\right|^{-\frac{s}{2}}\cdot\Psi\right) &\cong \mathrm{Hom}_{S_{2m}}\left(\pi\otimes\mathrm{ind}_{S_{2m}\cap P_{2m}}^{S_{2m}}\left(1\right),\left|\det\right|^{-\frac{s}{2}}\cdot\Psi\right)\\ &= \mathrm{Hom}_{S_{2m}}\left(\left(\left|\det\right|^{\frac{s}{2}}\cdot\Psi^{-1}\right)\cdot\pi\otimes\mathrm{ind}_{S_{2m}\cap P_{2m}}^{S_{2m}}\left(1\right),1\right)\\ &\cong \mathrm{Hom}_{S_{2m}}\left(\left(\left|\det\right|^{\frac{s}{2}}\cdot\Psi^{-1}\right)\cdot\pi,\widetilde{\mathrm{ind}_{S_{2m}\cap P_{2m}}^{S_{2m}}\left(1\right)}\right)\\ &\cong \mathrm{Hom}_{S_{2m}}\left(\left(\left|\det\right|^{\frac{s}{2}}\cdot\Psi^{-1}\right)\cdot\pi,\mathrm{Ind}_{S_{2m}\cap P_{2m}}^{S_{2m}}\left(\delta_{S_{2m}\cap P_{2m}\backslash S_{2m}}\right)\right).\end{aligned}$$

Here $\delta_{S_{2m}\cap P_{2m}\backslash S_{2m}}\left(p\right)=\frac{\delta_{S_{2m}\cap P_{2m}}(p)}{\delta_{S_{2m}}(p)}=\left|\det p\right|^{\frac{1}{2}}$, for $p\in S_{2m}\cap P_{2m}$, and we get

$$\begin{aligned}\mathrm{Hom}_{S_{2m}}\left(\left(\left|\det\right|^{\frac{s}{2}}\cdot\Psi^{-1}\right)\cdot\pi,\mathrm{Ind}_{S_{2m}\cap P_{2m}}^{S_{2m}}\left(\left|\det\right|^{\frac{1}{2}}\right)\right) &= \mathrm{Hom}_{S_{2m}\cap P_{2m}}\left(\left(\left|\det\right|^{\frac{s}{2}}\cdot\Psi^{-1}\right)\cdot\pi,\left|\det\right|^{\frac{1}{2}}\right)\\ &= \mathrm{Hom}_{S_{2m}\cap P_{2m}}\left(\left|\det\right|^{\frac{s-1}{2}}\pi,\Psi\right)\end{aligned}$$

By Corollary 3.52, $\mathrm{Hom}_{S_{2m}\cap P_{2m}}\left(\left|\det\right|^{\frac{s-1}{2}}\pi,\Psi\right)$ has dimension at most one, which implies that so does $\mathrm{Hom}_{S_{2m}}\left(\pi\otimes\mathcal{S}_0\left(F^m\right),\left|\det\right|^{-\frac{s}{2}}\cdot\Psi\right)$. Since $\mathrm{Hom}_{S_{2m}}\left(\pi\otimes\mathcal{S}\left(F^m\right),\left|\det\right|^{-\frac{s}{2}}\cdot\Psi\right)$ is embedded as a subspace of $\mathrm{Hom}_{S_{2m}}\left(\pi\otimes\mathcal{S}_0\left(F^m\right),\left|\det\right|^{-\frac{s}{2}}\cdot\Psi\right)$ for all values of q^{-s} except for a finite number of values, we get that for all values of q^{-s}, except for a finite number, $\mathrm{Hom}_{S_{2m}}\left(\pi\otimes\mathcal{S}\left(F^m\right),\left|\det\right|^{-\frac{s}{2}}\cdot\Psi\right)$ has dimension at most 1.

Recall that for a fixed value $s\in\mathbb{C}$, $B_s\left(W,\phi\right)=J_{\pi,\psi}\left(s,W,\phi\right)$ and $\tilde{B}_s\left(W,\phi\right)=\tilde{J}_{\pi,\psi}\left(s,W,\phi\right)$ are bilinear $\left|\det\right|^{-\frac{s}{2}}\cdot\Psi$-equivariant forms (Corollary 3.35), and therefore define elements of $\mathrm{Hom}_{S_{2m}}\left(\pi\otimes\mathcal{S}\left(F^m\right),\left|\det\right|^{-\frac{s}{2}}\cdot\Psi\right)$. Therefore, for every value of q^{-s}, except for a finite number of values, $\tilde{B}_s=\gamma_{\pi,\psi}\left(s\right)B_s$ where $\gamma_{\pi,\psi}\left(s\right)\in\mathbb{C}$. Choosing $W\in\mathcal{W}\left(\pi,\psi\right)$ and $\phi\in\mathcal{S}\left(F^m\right)$, such that $J_{\pi,\psi}\left(s,W,\phi\right)=1$ for every s, implies $\gamma_{\pi,\psi}\left(s\right)=\tilde{J}_{\pi,\psi}\left(s,W,\phi\right)$, for every value of q^{-s}, except for a finite number of values, which implies that $\gamma_{\pi,\psi}\left(s\right)$ is a rational function in the variable q^{-s}. For fixed $W\in\mathcal{W}\left(\pi,\psi\right)$ and $\phi\in\mathcal{S}\left(F^m\right)$, both sides of the equation $\tilde{J}_{\pi,\psi}\left(s,W,\phi\right)=\gamma_{\pi,\psi}\left(s\right)J_{\pi,\psi}\left(s,W,\phi\right)$ are rational functions in the variable q^{-s}. Since both sides agree for all but a finite number of values of q^{-s}, we get from the uniqueness theorem that they agree for all values of q^{-s}.

Finally, we write $\gamma_{\pi,\psi}\left(s\right)=\varepsilon_{\pi,\psi}\left(s\right)\cdot\frac{L\left(1-s,\tilde{\pi},\wedge^2\right)}{L\left(s,\pi,\wedge^2\right)}$ where $\varepsilon_{\pi,\psi}\left(s\right)\in\mathbb{C}\left(q^{-s}\right)$. We will show $\varepsilon_{\pi,\psi}\left(s\right)$ is an invertible element of $\mathbb{C}\left[q^s,q^{-s}\right]$. We have the following equation:

$$\frac{\tilde{J}_{\pi,\psi}\left(s,W,\phi\right)}{L\left(1-s,\tilde{\pi},\wedge^2\right)}=\varepsilon_{\pi,\psi}\left(s\right)\frac{J_{\pi,\psi}\left(s,W,\phi\right)}{L\left(s,\pi,\wedge^2\right)}.$$

Since $L\left(s,\pi,\wedge^2\right)$ is the generator of the fractional ideal $I_{\pi,\psi}$, there exists $\left(W_i\right)_{i=1}^N\subseteq\mathcal{W}\left(\pi,\psi\right)$, $\left(\phi_i\right)_{i=1}^N\subseteq\mathcal{S}\left(F^m\right)$, such that $\sum_{i=1}^N J_{\pi,\psi}\left(s,W_i,\phi_i\right)=L\left(s,\pi,\wedge^2\right)$. Substituting this in the equation yields $\varepsilon_{\pi,\psi}\left(s\right)=\sum_{i=1}^N\frac{\tilde{J}_{\pi,\psi}\left(s,W_i,\phi_i\right)}{L\left(1-s,\tilde{\pi},\wedge^2\right)}$, which implies that $\varepsilon_{\pi,\psi}$ is an element of $\mathbb{C}\left[q^s,q^{-s}\right]$. Likewise, one can choose $\left(W_i'\right)_{i=1}^{N'}\subseteq\mathcal{W}\left(\pi,\psi\right)$, $\left(\phi_i'\right)_{i=1}^{N'}\subseteq\mathcal{S}\left(F^m\right)$, such that $\sum_{i=1}^{N'}\tilde{J}_{\pi,\psi}\left(s,W_i',\phi_i'\right)=L\left(1-s,\tilde{\pi},\wedge^2\right)$. Substituting this in the equation yields $\varepsilon_{\pi,\psi}^{-1}\left(s\right)=\sum_{i=1}^{N'}\frac{J_{\pi,\psi}\left(s,W_i',\phi_i'\right)}{L\left(s,\pi,\wedge^2\right)}$, which implies $\varepsilon_{\pi,\psi}^{-1}(s) \in \mathbb{C}[q^s, q^{-s}]$. Therefore $\varepsilon_{\pi,\psi}(s)$ is an invertible element of $\mathbb{C}[q^s, q^{-s}]$, as requested. □

Remark 3.53. The calculations done in Subsection 1.2.3 yield that for $a \in F^*$,

$$\gamma_{\pi,\psi_a}(s) = \omega_\pi(a)^{2(m-1)} |a|^{2m(m-1)\left(s-\frac{1}{2}\right)} \gamma_{\pi,\psi}(s),$$

i.e. $\varepsilon_{\pi,\psi_a}(s) = \omega_\pi(a)^{2(m-1)} |a|^{2m(m-1)\left(s-\frac{1}{2}\right)} \varepsilon_{\pi,\psi}(s)$.

3.6. **Poles of the γ-factor, and Shalika functionals.** Let π be an irreducible supercuspidal representation of $\mathrm{GL}_{2m}(F)$. In this subsection we relate between a pole of the γ-factor of π and the existence of a Shalika functional. We begin with the following propositions which will be useful later.

Lemma 3.54. *Suppose that $J_{\pi,\psi}(s, W, \phi)$ has a pole at $s = 0$ for some $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in \mathcal{S}(F^m)$. Then $\omega_\pi \equiv 1$.*

Proof. Since π is supercuspidal, by Remark 3.33, $J_{\pi,\psi}(s, W, \phi) \in L(ms, \omega_\pi) \cdot \mathbb{C}[q^{-s}, q^s]$. Since $J_{\pi,\psi}(s, W, \phi)$ has a pole, this implies that ω_π is unramified, and then $L(ms, \omega_\pi) = \frac{1}{1-\omega_\pi(\varpi)q^{-ms}}$. Since $J_{\pi,\psi}(s, W, \phi)$ has a pole at $s = 0$, this implies $\omega_\pi(\varpi) = 1$, and therefore $\omega_\pi \equiv 1$. □

Definition 3.55. Suppose that $\omega_\pi \equiv 1$. We denote

$$l_{\pi,\psi}(W) = \int_{ZN\backslash G} \left(\int_{\mathcal{B}\backslash M} W\left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \right) \psi(-\mathrm{tr}(X))\, dX \right) dg.$$

This integral converges due to Proposition 3.28.

Proposition 3.56. *Suppose that $\omega_\pi \equiv 1$. Then for any $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in \mathcal{S}(F^m)$*

$$\lim_{s\to 0} \left(1 - q^{-ms}\right) J_\pi(s, W, \phi) = \phi(0) \cdot l_{\pi,\psi}(W)$$

Proof. We first consider two special cases.

If $\phi(0) = 0$, then by Remark 3.33, $J_{\pi,\psi}(s, W, \phi) \in \mathbb{C}[q^{-s}, q^s]$, and therefore

$$\lim_{s\to 0} \left(1 - q^{-ms}\right) J_\pi(s, W, \phi) = 0.$$

If $\phi = 1\chi_{\mathcal{O}^m}$, we have that $J_{\pi,\psi}(s, W, \phi)$ is equal to

$$\int_{A_{m-1}} da' \int_K dk \int_{\mathcal{B}\backslash M} dX \left(\delta_B^{-1}(a') W\left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} a'k & \\ & a'k \end{pmatrix} \right) \psi(-\mathrm{tr}X) \right) |\det(a')|^s$$

$$\cdot \int_{F^*} 1\chi_{\mathcal{O}^m}(\varepsilon a_m k) \underbrace{\omega_\pi(a_m)}_{=1} |a_m|^{ms}\, da_m.$$

Since $\varepsilon a_m k \in \mathcal{O}^m \iff |a_m| \le 1$, we get that

$$\int_{F^*} 1\chi_{\mathcal{O}^m}(\varepsilon a_m k) |a_m|^{ms}\, da_m = \sum_{i=0}^{\infty} \int_{\varpi^i \mathcal{O}^*} |a_m|^{ms}\, da_m = \frac{1}{1 - q^{-ms}}.$$

Therefore, we get that the limit $\lim_{s\to 0}(1 - q^{-ms}) J_{\pi,\psi}(s, W, \phi)$ is equal to

$$\int_{A_{m-1}} da' \int_K dk \int_{\mathcal{B}\backslash M} dX \left(\delta_B^{-1}(a') W\left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} a'k & \\ & a'k \end{pmatrix} \right) \psi(-\mathrm{tr}X) \right).$$

(Note that this value is finite by Proposition 3.28).

By the Iwasawa decomposition, this equals $l_{\pi,\psi}(W)$.

We move to the general case. Let $\phi \in \mathcal{S}(F^m)$. Write $\phi = \phi' + \phi(0) \cdot 1\chi_{\mathcal{O}^m}$, where $\phi' \in \mathcal{S}(F^m)$ with $\phi'(0) = 0$. Then

$$J_{\pi,\psi}(s, W, \phi) = J_{\pi,\psi}(s, W, \phi') + \phi(0) J_{\pi,\psi}(s, W, 1\chi_{\mathcal{O}^m}),$$

and from the previous two cases:

$$\lim_{s\to 0} \left(1 - q^{-ms}\right) J_{\pi,\psi}(s, W, \phi) = 0 + \phi(0) l_{\pi,\psi}(W).$$

□

Corollary 3.57. *Let $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in \mathcal{S}(F^m)$. Then $J_{\pi,\psi}(s, W, \phi)$ has a pole at $s = 0$ if and only if $\omega_\pi \equiv 1$ and $\phi(0) l_{\pi,\psi}(W) \neq 0$.*

Proof. First note that $\lim_{s\to 0} \frac{s}{1-q^{-ms}} = \frac{1}{m \log q} \neq 0$ and therefore $J_{\pi,\psi}(s, W, \phi)$ has a pole at $s = 0$ if and only if $\lim_{s\to 0}(1 - q^{-ms}) J_{\pi,\psi}(s, W, \phi) \neq 0$. The corollary now follows from Lemma 3.54 and Proposition 3.56. □

Corollary 3.58. *$L(s, \pi, \wedge^2)$ has a pole at $s = 0$ if and only if $\omega_\pi \equiv 1$ and there exists $W \in \mathcal{W}(\pi, \psi)$, such that $l_{\pi,\psi}(W) \neq 0$.*

Proof. $L(s, \pi, \wedge^2)$ has a pole at $s = 0$ if and only if one of the functions $J_{\pi,\psi}(s, W, \phi)$ has a pole at $s = 0$. The corollary now follows from the previous corollary. □

Theorem 3.59. *$\gamma_{\pi,\psi}(s)$ has a pole at $s = 1$ if and only if $\omega_\pi \equiv 1$ and there exists $W \in \mathcal{W}(\pi, \psi)$, such that $l_{\pi,\psi}(W) \neq 0$.*

Proof. Suppose that $\gamma_{\pi,\psi}(s)$ has a pole at $s = 1$. According to Theorem 3.31, there exists $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in \mathcal{S}(F^m)$, such that $J_{\pi,\psi}(s, W, \phi) = 1$. We substitute such W and ϕ in the functional equation to get $\gamma_{\pi,\psi}(s) = \tilde{J}_{\pi,\psi}(s, W, \phi)$. Recalling the definition of $\tilde{J}_{\pi,\psi}(s, W, \phi) = J_{\tilde{\pi},\theta^{-1}}\left(1 - s, W', \hat{\phi}\right)$ where $W' \in \mathcal{W}(\tilde{\pi}, \psi^{-1})$ is defined by

$$W'(g) = \tilde{\pi}\left(\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}\right) \tilde{W}(g) = W\left(w_{2m} g^l \begin{pmatrix} & I_m \\ I_m & \end{pmatrix}\right).$$

We get that $J_{\tilde{\pi},\theta^{-1}}\left(s, W', \hat{\phi}\right)$ has a pole at $s = 0$. According to Proposition 3.56 this implies that $\omega_{\tilde{\pi}} \equiv 1$ and $\hat{\phi}(0) l_{\tilde{\pi},\psi^{-1}}(W') \neq 0$, which implies that

$$l_{\tilde{\pi},\psi^{-1}}(W') = \int_{ZN\backslash G} \left(\int_{\mathcal{B}\backslash M} W\left(w_{2m} w_{m,m}^l \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}^l \begin{pmatrix} g & \\ & g \end{pmatrix}^l \begin{pmatrix} & I_m \\ I_m & \end{pmatrix}\right) \psi^{-1}(-\mathrm{tr}(X))\, dX\right) dg \neq 0.$$

Using the fact that w_{2m} and $w_{m,m}$ commute, and the same conjugation techniques as in Subsection 1.2.1, we get that $l_{\tilde{\pi},\psi^{-1}}(W') = l_{\pi,\psi}(W)$, and this direction is proved.

For the other direction, suppose that $\omega_{\tilde{\pi}} \equiv 1$, and that there exists $W \in \mathcal{W}(\pi, \psi)$, such that $l_{\pi,\psi}(W) \neq 0$. Again, we get that $0 \neq l_{\tilde{\pi},\psi^{-1}}(W') = l_{\pi,\psi}(W)$, where W' is defined as above. Since $I_{\pi,\psi} \subseteq L(ms, \omega_\pi) \mathbb{C}[q^{-s}, q^s]$, we have $L(s, \pi, \wedge^2) = \frac{1}{p_1(q^{-s})}$, $L(s, \tilde{\pi}, \wedge^2) = \frac{1}{p_2(q^{-s})}$, where $p_1(z), p_2(z) \in \mathbb{C}[z]$ are such that $p_1(0) = p_2(0) = 1$, $p_1(z), p_2(z) \mid 1 - z^m$, and such that $1-z \mid p_1(z), p_2(z)$ (as $L(s,\pi,\wedge^2)$, $L(s,\tilde{\pi},\wedge^2)$ have poles at $s=0$ from the previous corollary) and therefore

$$\gamma_{\pi,\psi}(s) = \varepsilon_{\pi,\psi}(s) \cdot \frac{p_1(q^{-s})}{p_2(q^{-(1-s)})},$$

where $\varepsilon_{\pi,\psi}(s) = c \cdot q^{ks}$, $c \in \mathbb{C}^*$, $k \in \mathbb{Z}$.

Since $p_1(q^{-1}) \neq 0$ and $p_2(1) = 0$, it is clear that $\gamma_{\pi,\psi}$ has a pole at $s=1$. □

Definition 3.60. A functional $l : \mathcal{W}(\pi,\psi) \to \mathbb{C}$ is called a Shalika functional if for every $W \in \mathcal{W}(\pi,\psi)$ and $\left(\begin{smallmatrix} g & X \\ & g \end{smallmatrix}\right) \in S_{2m}$, one has $l\left(\pi\left(\begin{smallmatrix} g & X \\ & g \end{smallmatrix}\right) W\right) = \psi(\mathrm{tr}(g^{-1}X))\, l(W)$.

Proposition 3.61. *Suppose that $\omega_\pi \equiv 1$, then the functional $l_{\pi,\psi}$ defined above is a Shalika functional.*

Proof. This follows directly by changing variables in the integral defining $l_{\pi,\psi}$, just as in the proof of the equivariance properties of $J_{\pi,\psi}$ (Proposition 1.10). □

We conclude this subsection with a theorem.

Theorem 3.62. *Let π be an irreducible supercuspidal representation of $\mathrm{GL}_{2m}(F)$. The following are equivalent:*

(1) $\omega_\pi \equiv 1$ *and* $l_{\pi,\psi} \not\equiv 0$.
(2) $\gamma_{\pi,\psi}(s)$ *has a pole at* $s=1$.
(3) $L(s,\pi,\wedge^2)$ *has a pole at* $s=0$.

3.7. **The local exterior square L function for supercuspidal representations.** Let π be an irreducible supercuspidal representation of $\mathrm{GL}_{2m}(F)$. In this subsection, we give an explicit expression for $L(s,\pi,\wedge^2)$ (See Remark 3.31 for the definition).

Proposition 3.63. *Suppose that ω_π is ramified, i.e. $\omega_\pi \restriction_{\mathcal{O}^*} \not\equiv 1$. Then $L(s,\pi,\wedge^2) = 1$.*

Proof. The inclusion $I_{\pi,\psi} \supseteq \mathbb{C}[q^{-s}, q^s]$ is always true (Theorem 3.31).

Regarding the inclusion $I_{\pi,\psi} \subseteq \mathbb{C}[q^{-s}, q^s]$, from Remark 3.33, we have $I_{\pi,\psi} \subseteq L(ms,\omega_\pi)\,\mathbb{C}[q^{-s}, q^s]$. Since ω_π is unramified, it follows from Theorem 3.3 that $L(ms,\omega_\pi) = 1$, and the proposition follows. □

Proposition 3.64. *Suppose that $\omega_\pi \equiv 1$. Let $\zeta = e^{\frac{2\pi i}{m}}$. Then*

$$L(s,\pi,\wedge^2) = \prod_{k \in S_{\pi,\psi}} \frac{1}{1-\zeta^k q^{-s}},$$

where

$$S_{\pi,\psi} = \Big\{ 0 \le k \le m-1 \mid \exists W \in \mathcal{W}(\pi,\psi), \int_{ZN\backslash G} \left(\int_{\mathcal{B}\backslash M} W\left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \right) \psi(-\mathrm{tr}(X))\, dX \right) \left|\det g\right|^{\frac{2\pi i k}{m \log q}} dg \neq 0 \Big\}.$$

Proof. Since $J_{\pi,\psi}\left(s + \frac{2\pi i k}{m \log q}, W, \phi\right) = J_{\pi \cdot |\det|^{\frac{\pi i k}{m \log q}},\psi}(s, W, \phi)$, we get from Proposition 3.57, that $J_{\pi,\psi}\left(s + \frac{2\pi i k}{m \log q}, W, \phi\right)$ has a pole at $s=0$, if and only if there exists $W \in \mathcal{W}(\pi,\psi)$, such that $l_{\pi \cdot |\det|^{\frac{\pi i k}{m \log q}},\psi}(W) \neq 0$. This is equivalent to $k \in S_{\pi,\psi}$.

Since $L(s,\pi,\wedge^2) = \frac{1}{p(q^{-s})}$, where $p(z) \mid (1-z^m)$ (since $I_{\pi,\psi} \subseteq L(ms,\omega_\pi)\mathbb{C}[q^{-s},q^s]$), we get that $p(z) = \prod_{k\in S_{\pi,\psi}}\left(1-\zeta^k z\right)$, as required. $\square$

We now move to the case where ω_π is an unramified character. Suppose that ω_π is a general unramified character. For $z \in F^*$ write $z = \varpi^k \cdot u$, where $|u| = 1$, $k \in \mathbb{Z}$. Then, $\omega_\pi(z) = \omega_\pi(\varpi)^k$. Therefore, we can write $\omega_\pi(z) = |z|^{s_0}$, where $s_0 = \frac{\log \omega_\pi(\varpi)}{\log q}$. Consider the representation $\pi' = \pi \cdot |\det|^{-\frac{s_0}{2m}}$. π' is irreducible and supercuspidal with a trivial central character. Therefore, from Proposition 3.64,

$$L\left(s,\pi',\wedge^2\right) = \prod_{k\in S_{\pi',\psi}} \frac{1}{1-\zeta^k q^{-s}}.$$

As in the proof of Theorem 3.23, $J_{\pi',\psi}\left(s+\frac{s_0}{m},W,\phi\right) = J_{\pi,\psi}(s,W,\phi)$, and therefore it follows that $L\left(s+\frac{s_0}{m},\pi',\wedge^2\right) = L(s,\pi,\wedge^2)$. Therefore

$$L\left(s,\pi,\wedge^2\right) = \prod_{k\in S_{\pi,\psi}} \frac{1}{1-\omega_\pi(\varpi)^{\frac{1}{m}}\zeta^k q^{-s}},$$

where

$$S_{\pi,\psi} = \Big\{0 \le k \le m-1 \mid \exists W \in \mathcal{W}(\pi,\psi),$$
$$\int_{ZN\backslash G}\left(\int_{\mathcal{B}\backslash M} W\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m\end{pmatrix}\begin{pmatrix} g & \\ & g\end{pmatrix}\right)\psi(-\mathrm{tr}(X))\,dX\right)|\det g|^{\frac{2\pi i k-\log\omega_\pi(\varpi)}{m\log q}}\,dg \neq 0\Big\}.$$

Theorem 3.65. *Let π be an irreducible supercuspidal representation of $\mathrm{GL}_{2m}(F)$. If ω_π is ramified, then $L(s,\pi,\wedge^2) = L(ms,\omega_\pi) = 1$. If ω_π is unramified then*

$$L\left(s,\pi,\wedge^2\right) = \prod_{k\in S_{\pi,\psi}} \frac{1}{1-\omega_\pi(\varpi)^{\frac{1}{m}}\zeta^k q^{-s}},$$

where

$$S_{\pi,\psi} = \Big\{0 \le k \le m-1 \mid \exists W \in \mathcal{W}(\pi,\psi),$$
$$\int_{ZN\backslash G}\left(\int_{\mathcal{B}\backslash M} W\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m\end{pmatrix}\begin{pmatrix} g & \\ & g\end{pmatrix}\right)\psi(-\mathrm{tr}(X))\,dX\right)|\det g|^{\frac{2\pi i k-\log\omega_\pi(\varpi)}{m\log q}}\,dg \neq 0\Big\}.$$

4. Level zero representations

Towards this section, F is again a p-adic field with absolute value $|\cdot|$, $\mathcal{O}$ denotes the ring of integers of F, $\mathcal{P}$ denotes the unique prime ideal of $\mathcal{O}$, ϖ is a uniformizer of $\mathcal{O}$ (a generator of $\mathcal{P}$), $q = \left|{}^{\mathcal{O}}/_{\mathcal{P}}\right|$. Then ${}^{\mathcal{O}}/_{\mathcal{P}} \cong \mathbb{F}_q$.

We denote by ν the quotient map $\nu : \mathcal{O} \to \mathbb{F}_q$. ν defines a homomorphism $\nu : \mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(\mathbb{F}_q)$.

4.1. **Preliminaries.**

4.1.1. *Level zero representations.* Let n be a positive integer.

Let (π_0, V_0) be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$. We describe a method to construct an irreducible supercuspidal representation (π, V) of $\mathrm{GL}_n(F)$.

Using ν and π_0, we can define a representation (π_0', V_0) of $\mathrm{GL}_n(\mathcal{O})$ by $\pi_0'(k) = \pi_0(\nu(k))$, for $k \in \mathrm{GL}_n(\mathcal{O})$.

Let $\chi : F^* \to \mathbb{C}^*$ be a character of F^*, such that $\chi \restriction_{\mathcal{O}^*} = \omega_{\pi_0} \circ \nu \restriction_{\mathcal{O}^*}$, where ω_{π_0} is the central character of π_0. Such characters exist: using the decomposition $F^* = \langle \varpi \rangle \times \mathcal{O}^*$, one sees that such characters are exactly the characters of the form $\chi_{z_0}\left(\varpi^k \cdot u\right) = z_0^k \cdot \omega_{\pi_0}(\nu(u))$, where $z_0 \in \mathbb{C}^*$ ($u \in \mathcal{O}^*$, $k \in \mathbb{Z}$).

We define a representation $(\chi\pi_0', V_0)$ of $F^* \cdot \mathrm{GL}_n(\mathcal{O})$ by $(\chi\pi_0')(z \cdot k) = \chi(z) \cdot \pi_0(\nu(k))$, where $z \in F^*$ and $k \in \mathrm{GL}_n(\mathcal{O})$. It is easy to check that $\chi\pi_0'$ is well defined. Since $\mathrm{GL}_n(\mathcal{O})$ is an open subgroup, it follows that $F^* \cdot \mathrm{GL}_n(\mathcal{O})$ is an open subgroup, and therefore $F^* \cdot \mathrm{GL}_n(\mathcal{O})$ is also a closed subgroup.

We define $(\pi, V) = \mathrm{ind}_{F^* \cdot \mathrm{GL}_n(\mathcal{O})}^{\mathrm{GL}_n(F)}(\chi\pi_0')$.

Theorem 4.1. *(π, V) is an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$.* [PR08, Theorem 6.2]

Representations obtained through this method are called irreducible level zero (or depth zero) supercuspidal representations of $\mathrm{GL}_n(F)$.

4.1.2. *Whittaker model lift.* Let (π_0, V_0) be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$, and let (π, V) be a level zero representation, constructed through π_0, with respect to the character $\chi : F^* \to \mathbb{C}^*$. In this subsection, we relate between the Whittaker models of π and π_0.

Let $\psi : F \to \mathbb{C}^*$ be a non-trivial character, such that its conductor is $\mathcal{P}$ (i.e. $\psi \restriction_{\mathcal{P}} \equiv 1$ and $\psi \restriction_{\mathcal{O}} \not\equiv 1$). We denote by $\psi_0 : \mathbb{F}_q \to \mathbb{C}^*$ the character defined by $\psi_0(x_0) = \psi(x)$, where $x_0 \in \mathbb{F}_q$ and $x \in \mathcal{O}$ with $\nu(x) = x_0$. ψ_0 is well defined, as $\psi \restriction_{\mathcal{P}} \equiv 1$, and ψ_0 is non-trivial, as $\psi \restriction_{\mathcal{O}} \not\equiv 1$.

As noted in Subsection 1.1.1, π_0 is generic.

Let $0 \neq T_0 \in \mathrm{Hom}_{N_n(\mathbb{F}_q)}\left(\pi_0 \restriction_{N_n(\mathbb{F}_q)}, \psi_0\right)$ be a non-zero Whittaker functional of π_0 with respect to ψ_0.

We give a description of the Whittaker model $\mathcal{W}(\pi, \psi)$ using T_0.

We start with a useful Lemma:

Lemma 4.2. $N_n \cap (F^* \cdot \mathrm{GL}_n(\mathcal{O})) = N_n(\mathcal{O})$, *where $N_n \subseteq \mathrm{GL}_n(F)$ is the upper triangular unipotent matrix subgroup and $N_n(\mathcal{O}) = N_n \cap \mathrm{GL}_n(\mathcal{O})$.*

Proof. For the inclusion $N_n \cap (F^* \cdot \mathrm{GL}_n(\mathcal{O})) \subseteq N_n(\mathcal{O})$, suppose that $u = z \cdot k$, where $u \in N_n$, $z \in F^*$ and $k \in \mathrm{GL}_n(\mathcal{O})$. Taking the determinant of both sides yields $|z|^n = 1$, and therefore $|z| = 1$, which implies $z \in \mathcal{O}^*$. Therefore $u \in \mathrm{GL}_n(\mathcal{O}) \cap N_n = N_n(\mathcal{O})$.

The other inclusion is trivial. $\square$

Theorem 4.3. *The functional $T : V \to \mathbb{C}$ defined by*

$$\langle T, f \rangle = \int_{N_n(\mathcal{O}) \backslash N_n} \psi^{-1}(u) \langle T_0, f(u) \rangle \, du \qquad (f \in V)$$

is a non-zero Whittaker functional $T \in \mathrm{Hom}_{N_n}(\pi \restriction_{N_n}, \psi)$.

Proof. The integrand is well defined: for $k \in N_n(\mathcal{O})$, $f(ku) = \pi_0(\nu(k)) f(u)$, and therefore $\psi^{-1}(ku) \langle T_0, f(ku) \rangle = \psi^{-1}(ku) \underbrace{\psi_0(\nu(k))}_{\psi(k)} \langle T_0, f(u) \rangle$.

The integral converges: since $f \in \mathrm{ind}_{F^* \cdot \mathrm{GL}_n(\mathcal{O})}^{\mathrm{GL}_n(F)}(\chi \pi_0)$, there exists a compact subset $C \subseteq \mathrm{GL}_n(F)$, such that $\mathrm{supp} f \subseteq (F^* \cdot \mathrm{GL}_n(\mathcal{O})) \cdot C$. Therefore the integral is integrated on cosets of the form $N_n(\mathcal{O}) u$, where $u \in N_n \cap (F^* \cdot \mathrm{GL}_n(\mathcal{O}) \cdot C)$. Suppose that $u = zkc$, where $u \in N_n(F)$, $z \in F^*$, $k \in \mathrm{GL}_n(\mathcal{O})$ and $c \in C$. Then $zI_n = uc^{-1}k^{-1} \in N_n \cdot C^{-1} \cdot \mathrm{GL}_n(\mathcal{O})$. By comparing determinants we get that $z^n \in \det(C^{-1}) \cdot \mathcal{O}^*$, and therefore $|z|^n \in |\det(C^{-1})|$. C^{-1} is compact, and therefore $|z|$ is bounded, i.e. z belongs to a compact set $C_Z \subseteq F^*$, and $u \in C_Z \cdot \mathrm{GL}_n(\mathcal{O}) \cdot C$ belongs to a compact set. Therefore, the integral is integrated on a compact subset of $N_n(\mathcal{O}) \backslash N_n$, and therefore converges.

It is clear by its definition that $T \in \mathrm{Hom}_{N_n}(\pi \restriction_{N_n}, \psi)$. We show it is not identically zero.

Let $v_0 \in V_0$ such that $\langle T_0, v_0 \rangle \neq 0$. We define $f_{v_0} \in V$ by

$$f_{v_0}(g) = \begin{cases} \chi(z) \pi_0(\nu(k)) v_0 & g = zk, \, z \in F^*, \, k \in \mathrm{GL}_n(\mathcal{O}) \\ 0 & \text{otherwise} \end{cases},$$

then $f_{v_0} \in \mathrm{ind}_{F^* \cdot \mathrm{GL}_n(\mathcal{O})}^{\mathrm{GL}_n(F)}(\chi \cdot \pi_0')$. We have

$$\langle T, f_{v_0} \rangle = \int_{N_n(\mathcal{O}) \backslash N_n} \psi^{-1}(u) \langle T_0, f_{v_0}(u) \rangle \, du,$$

and u is integrated only on cosets of the form $N_n \cap (F^* \cdot \mathrm{GL}_n(\mathcal{O})) = N_n(\mathcal{O})$. This implies that the value of the integral equals $\langle T_0, f_{v_0}(I_n) \rangle = \langle T_0, v_0 \rangle \neq 0$. $\square$

We now express the Whittaker model $\mathcal{W}(\pi, \psi)$ using Frobenius reciprocity: for $f \in V$ we denote by $W_f : \mathrm{GL}_n(F) \to \mathbb{C}$ the function $W_f(g) = \langle T, \pi(g) f \rangle$. Then

$$\mathcal{W}(\pi, \psi) = \{ W_f \mid f \in V \}.$$

We also denote for $v_0 \in V_0$ the function $W_{v_0}^0 : \mathrm{GL}_n(\mathbb{F}_q) \to \mathbb{C}$, defined by $W_{v_0}^0(g) = \langle T_0, \pi_0(g) v_0 \rangle$. Then

$$\mathcal{W}(\pi_0, \psi_0) = \left\{ W_{v_0}^0 \mid v_0 \in V_0 \right\}.$$

We will be interested in elements of the form W_f for $f = f_{v_0}$, for $v_0 \in V_0$, as above:

$$f_{v_0}(g) = \begin{cases} \chi(z) \pi_0(\nu(k)) v_0 & g = zk, \, z \in F^*, \, k \in \mathrm{GL}_n(\mathcal{O}) \\ 0 & \text{otherwise} \end{cases}.$$

It is clear that $f_{v_0} \in \mathrm{ind}_{F^* \cdot \mathrm{GL}_n(\mathcal{O})}^{\mathrm{GL}_n(F)} (\chi \cdot \pi_0')$. We denote $W_{f_{v_0}} = W_{v_0}$.

Proposition 4.4. $\mathrm{supp} W_{v_0} \subseteq N_n \cdot F^* \cdot \mathrm{GL}_n(\mathcal{O})$. *For* $u_0 \in N_n$, $z \in F^*$, $k \in \mathrm{GL}_n(\mathcal{O})$ *we have*

$$W_{v_0}(u_0 z k) = \psi(u_0) \chi(z) W_{v_0}^0(\nu(k)).$$

Proof. We write

$$W_{v_0}(g) = \int_{N_n(\mathcal{O}) \backslash N_n} \psi^{-1}(u) \langle T_0, f_{v_0}(ug) \rangle \, du.$$

Suppose that $g \in \mathrm{supp} W_{v_0}$. Then $u_0 g \in \mathrm{supp} f_{v_0} = F^* \cdot \mathrm{GL}_n(\mathcal{O})$, for some $u_0 \in N_n$. It is now clear that $g \in N_n \cdot F^* \cdot \mathrm{GL}_n(\mathcal{O})$.

Let $z \in F^*$ and $k \in \mathrm{GL}_n(\mathcal{O})$. Then

$$W_{v_0}(zk) = \chi(z) \int_{N_n(\mathcal{O}) \backslash N_n} \psi^{-1}(u) \langle T_0, f_{v_0}(uk) \rangle \, du.$$

Suppose that $u \in N_n$ such that $uk \in \mathrm{supp} f_{v_0} = F^* \cdot \mathrm{GL}_n(\mathcal{O})$. Then $u \in (F^* \cdot \mathrm{GL}_n(\mathcal{O})) \cap N_n = N_n(\mathcal{O})$. Therefore the integral is integrated on the single coset I_n, and results with the value

$$W_{v_0}(zk) = \chi(z) \underbrace{\langle T_0, \pi_0(\nu(k)) v_0 \rangle}_{W_{v_0}^0(\nu(k))}.$$

Since $W_{v_0} \in \mathcal{W}(\pi, \psi)$, we have $W_{v_0}(u_0 z k) = \psi(u_0) W_{v_0}(zk)$, and we get the required result. □

4.1.3. *Lifted Schwartz functions.* We will be interested in Schwartz functions obtained in the following fashion: Let ϕ be a function $\phi : \mathbb{F}_q^m \to \mathbb{C}$. We define a function on F^m, denoted by F_ϕ by

$$F_\phi(x) = \begin{cases} \phi(\nu(x)) & x \in \mathcal{O}^m \\ 0 & x \notin \mathcal{O}^m \end{cases}.$$

It is clear that F_ϕ is a Schwartz function which is invariant to translations of $\mathcal{P}^m$.

Fix a non-trivial character $\psi^{\mathcal{F}} : F \to \mathbb{C}^*$ whose conductor is $\mathcal{P}$.

Proposition 4.5. *Let* $\widehat{F_\phi}$ *be the Fourier transform of* F_ϕ *with respect to* $\psi^{\mathcal{F}}$, *defined as* $\widehat{F_\phi}(y) = \int_{F^m} F_\phi(x) \psi^{\mathcal{F}}(\langle x, y \rangle) \, dx$. *Then*

$$\widehat{F_\phi}(y) = F_{\hat{\phi}}(y),$$

where $\hat{\phi}(x) = \frac{1}{q^m} \sum_{a \in \mathbb{F}_q^m} \phi(a) \psi_0^{\mathcal{F}}(\langle a, x \rangle)$.

Proof. We begin with some properties of the Fourier transform: for a Schwartz function $f : F^m \to \mathbb{C}$ and $a \in F^m$, $b \in F^*$, we denote $f_{a,b}(x) = f(a + bx)$. A direct computation shows that

$$\widehat{f_{a,b}}(y) = \frac{1}{|b|^m} \psi^{\mathcal{F}}\left(\left\langle -\frac{a}{b}, y \right\rangle\right) \hat{f}\left(\frac{y}{b}\right).$$

Next we compute the Fourier transform of $1\chi_{\mathcal{O}^m}$:

$$\widehat{1\chi_{\mathcal{O}^m}}(y) = \int_{\mathcal{O}^m} \psi^{\mathcal{F}}(\langle x, y\rangle)\, dx.$$

For $y \in \mathcal{O}^m - \mathcal{P}^m$, the character $x \mapsto \psi^{\mathcal{F}}(\langle x, y\rangle)$ is non-trivial (since the conductor of ψ is $\mathcal{P}$), and therefore $\widehat{1\chi_{\mathcal{O}}}(y) = 0$ for such y. For $y \in \mathcal{P}^m$, the character $x \mapsto \psi^{\mathcal{F}}(\langle x, y\rangle)$ is trivial, and therefore $\widehat{1\chi_{\mathcal{O}}}(y) = 1$ for such y. Therefore we have $\widehat{1\chi_{\mathcal{O}^m}}(y) = 1\chi_{\mathcal{P}^m}(y)$.

Finally, let $\phi : \mathbb{F}_q^m \to \mathbb{C}$. Then $F_\phi = \sum_{a\in\mathbb{F}_q^m} 1\chi_{a'+\mathcal{P}^m} \cdot \phi(a)$ where for every $a \in \mathbb{F}_q^m$, $a' \in \mathcal{O}^m$ is an element, such that $\nu(a') = a$. A direct computation shows that

$$1\chi_{a'+\mathcal{P}^m} = (1\chi_{\mathcal{O}^m})_{-\frac{a'}{\varpi}, \frac{1}{\varpi}}.$$

Therefore

$$F_\phi = \sum_{a\in\mathbb{F}_q^m} \phi(a) \cdot (1\chi_{\mathcal{O}^m})_{-\frac{a'}{\varpi}, \frac{1}{\varpi}}.$$

Applying the above properties of the Fourier transform, we get

$$\widehat{F_\phi}(y) = \sum_{a\in\mathbb{F}_q^m} \phi(a) \cdot \frac{1}{|\varpi^{-1}|^m} \psi^{\mathcal{F}}(\langle a', y\rangle)\, 1\chi_{\mathcal{P}^m}(\varpi y).$$

Since $|\varpi^{-1}| = q$ and $1\chi_{\mathcal{P}^m}(\varpi y) = 1\chi_{\mathcal{O}^m}(y)$, we get that

$$\widehat{F_\phi}(y) = \frac{1}{q^m} \sum_{a\in\mathbb{F}_q^m} \phi(a)\, \psi^{\mathcal{F}}(\langle a', y\rangle)\, 1\chi_{\mathcal{O}^m}(y).$$

For $y \notin \mathcal{O}^m$, we have that $\widehat{F_\phi}(y) = 0$. Suppose $y \in \mathcal{O}^m$, then since $\psi^{\mathcal{F}} \restriction_{\mathcal{P}} \equiv 1$, we have $\psi^{\mathcal{F}}(\langle a', y\rangle) = \psi_0^{\mathcal{F}}(\langle a, \nu(y)\rangle)$, and therefore $\widehat{F_\phi}(y) = \hat{\phi}(\nu(y))$. We conclude that $\widehat{F_\phi} = F_{\hat{\phi}}$, as required. □

4.2. **The Jacquet Shalika integral of a level zero supercuspidal representation.** Let m be a positive integer. Let (π_0, V_0) be an irreducible cuspidal representation of $\mathrm{GL}_{2m}(\mathbb{F}_q)$, and let (π, V) be a level zero representation, constructed through π_0, with respect to the central character $\chi : F^* \to \mathbb{C}$. In this subsection, we relate between the integrals J_{π_0,ψ_0} and $J_{\pi,\psi}$.

Remark 4.6. Suppose that π' is the level zero representation, constructed through π_0, with respect to the central character χ', which is obtained by defining $\chi'(\varpi) = 1$. Let $s_0 \in \mathbb{C}$, such that $\chi(\varpi) = q^{-s_0}$. Then $\pi = \pi' \cdot |\det|^{\frac{s_0}{2m}}$, and for every $s \in \mathbb{C}$, $v \in V_0$, $\phi \in \mathcal{S}(\mathbb{F}_q^m)$, we have

$$J_{\pi,\psi}(s, W_v, F_\phi) = J_{\pi',\psi}\left(s + \frac{s_0}{m}, W_v, F_\phi\right),$$
$$\tilde{J}_{\pi,\psi}(s, W_v, F_\phi) = \tilde{J}_{\pi',\psi}\left(s + \frac{s_0}{m}, W_v, F_\phi\right),$$

i.e. the choice of $\chi(\varpi)$ only affects $J_{\pi,\psi}$, $\tilde{J}_{\pi,\psi}$ (and therefore also $\gamma_{\pi,\psi}$, $\varepsilon_{\pi,\psi}$, $L(s,\pi,\wedge^2)$, $L(1-s,\tilde{\pi},\wedge^2)$) by a translation by $\frac{s_0}{m}$.

Proposition 4.7. *There exists a choice of the Haar measures $\mu_{\mathcal{B}(F)\backslash M_m(F)}$, $\mu_{N_m(F)\backslash \mathrm{GL}_m(F)}$, such that for any $\phi : \mathbb{F}_q^m \to \mathbb{C}$ and $v \in V_0$, one has*

$$J_{\pi,\psi}(s, W_v, F_\phi) = J_{\pi_0,\psi_0}\left(W_v^0, \phi\right) + J_{\pi_0,\psi_0}\left(W_v^0, 1\right) \cdot \frac{\phi(0)\chi(\varpi)\cdot q^{-ms}}{1-\chi(\varpi)\cdot q^{-ms}}.$$

Proof. Using the same steps as in the proof of Theorem 3.23, we have

(4.1)

$$J_{\pi,\psi}(s, W_v, F_\phi) = \int_{A_{m-1}} da' \int_K dk \int_{\mathcal{N}^-} dX \left(\delta_B^{-1}(a') W_v\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m\end{pmatrix}\begin{pmatrix} a'k & \\ & a'k\end{pmatrix}\right)\right)|\det(a')|^s \cdot$$
$$\cdot \int_{F^*} F_\phi(\varepsilon a_m k)|a_m|^{ms}\omega_\pi(a_m)\, da_m.$$

We will show that a' is integrated on $(\mathcal{O}^*)^{m-1}$, and that X is integrated on $\mathcal{N}^- \cap M_m(\mathcal{O})$. Then we will be able to use Proposition 4.4.

Continuing, following the steps of Theorem 3.23, we get that

$$J_{\pi,\psi}(s, W_v, F_\phi) = \int_{A_{m-1}} da' \int_K dk \int_{\mathcal{N}^-} dZ \left(\delta_B^{-2}(a')\psi\left(bn_Zb^{-1}\right) W_v\left(bt_Zk_Zw_{m,m}\begin{pmatrix} k & \\ & k\end{pmatrix}\right)\right)|\det(a')|^s$$
$$\cdot \int_{F^*} F_\phi(\varepsilon a_m k)|a_m|^{ms}\omega_\pi(a_m)\, da_m,$$

where $Z = a'^{-1}Xa'$, $b = \mathrm{diag}\left(a'_1, a'_1, a'_2, a'_2, \dots, a'_{m-1}, a'_{m-1}, 1, 1\right)$, $u_Z = \left(\begin{smallmatrix} I_m & Z \\ & I_m\end{smallmatrix}\right)$ and $u_Z = n_Zt_Zk_Z$ is an Iwasawa decomposition of u_Z as in Proposition 3.22.

Suppose that $bt_Zk_Zw_{m,m}\left(\begin{smallmatrix} k & \\ & k\end{smallmatrix}\right) \in \mathrm{supp}W_v$, then by Proposition 4.4, $bt_Zk_Zw_{m,m}\left(\begin{smallmatrix} k & \\ & k\end{smallmatrix}\right) \in N_{2m}\cdot F^*\cdot K_{2m}$ (where $N_{2m} \subseteq \mathrm{GL}_m(F)$ is the upper triangular unipotent matrix subgroup and $K_{2m} = \mathrm{GL}_{2m}(\mathcal{O})$), and therefore $bt_Z = u(\lambda I_{2m})k$, where $u \in N_{2m}$, $\lambda \in F^*$ and $k \in K_{2m}$. The equality $u^{-1}bt_Z(\lambda^{-1}I_{2m}) = k$ implies that k is an upper triangular matrix, and therefore all of its diagonal elements are of absolute value one. Since the last diagonal element of both b and t_Z equals 1, this implies that $|\lambda| = 1$. Therefore the diagonal of $u(\lambda I_{2m})k$ consists of elements having absolute value one, and thus so does the diagonal of bt_Z. Writing $t = \mathrm{diag}(t_1, t_2, \dots t_{2m-1}, 1)$, we get that $|a'_i \cdot t_{2i-1}| = 1$ and $|a'_i \cdot t_{2i}| = 1$ for $1 \le i \le m-1$ and $|t_{2i-1}| = 1$. Therefore $|t_{2i}| = |t_{2i-1}|$ for $1 \le i \le m-1$. By Theorem 3.15, $|t_{2i}| \le 1$ and $|t_{2i-1}| \ge 1$, and therefore we get that $|t_i| = 1$, for every $1 \le i \le 2m-1$, which implies that $|a'_i| = 1$, for every $1 \le i \le 2m-1$. By Proposition 3.21, we have $\|Z\|^{\frac{1}{2m}} \le \prod_{\substack{1\le i\le 2m \\ i \text{ is odd}}}|t_i| = 1$, and therefore $Z \in M_m(\mathcal{O})$. This implies that $X = a'Za'^{-1} \in M_m(\mathcal{O})$.

We therefore have that a' is integrated on $(\mathcal{O}^*)^{m-1}$, and that X is integrated on $\mathcal{N}^-(\mathcal{O})$, where $(\mathcal{O}^*)^{m-1}$ is realized with the diagonal matrices consisting of elements from $\mathcal{O}^*$, and $\mathcal{N}^-(\mathcal{O})$ is the lower triangular nilpotent matrix subgroup of $M_m(\mathcal{O})$. Since $a' \in (\mathcal{O}^*)^{m-1}$, $\delta_B^{-1}(a') = 1$. Replacing $k = a'^{-1}k'$ in (4.1) yields

$$J_{\pi,\psi}(s, W_v, F_\phi) = \int_{(\mathcal{O}^*)^{m-1}} da' \int_K dk' \int_{\mathcal{N}^-(\mathcal{O})} dX \left(W_v\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m\end{pmatrix}\begin{pmatrix} k' & \\ & k'\end{pmatrix}\right)\right)\cdot$$
$$\cdot \int_{F^*} F_\phi\left(\varepsilon a_m\left(a'^{-1}k'\right)\right)|a_m|^{ms}\chi(a_m)\, da_m.$$

Note that since $a'^{-1} \in A_{m-1}$, its last row equals ε, and therefore $\varepsilon a_m a'^{-1} k' = \varepsilon a_m k$, and we are left with the following integral:

$$J_{\pi,\psi}(s, W_v, F_\phi) = \int_K dk' \int_{\mathcal{N}^-(\mathcal{O})} dX \left(W_v \left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} k' & \\ & k' \end{pmatrix} \right) \right) \cdot$$
$$\cdot \int_{F^*} F_\phi(\varepsilon a_m k') |a_m|^{ms} \chi(a_m)\, da_m.$$

We consider the following integral for a fixed $k' \in \mathrm{GL}_m(\mathcal{O})$

$$\int_{F^*} F_\phi(\varepsilon a_m k') |a_m|^{ms} \chi(a_m)\, da_m = \sum_{i=-\infty}^{\infty} \chi(\varpi)^i q^{-ims} \int_{\mathcal{O}^*} F_\phi\left(\varepsilon \varpi^i a_m k'\right) \chi(a_m)\, da_m.$$

For $i < 0$, $\varepsilon \varpi^i a_m k' \notin \mathcal{O}^m$ for any $a_m \in \mathcal{O}^*$, and therefore $F_\phi(\varepsilon \varpi^i a_m k') = 0$. For $i \geq 1$, $\varepsilon \varpi^i a_m k' \in \mathcal{P}^m$, and therefore $F_\phi(\varepsilon \varpi^i a_m k') = \phi(0)$ and

$$\sum_{i=1}^{\infty} \chi(\varpi)^i q^{-ims} \int_{\mathcal{O}^*} F_\phi\left(\varepsilon \varpi^i a_m k'\right) \chi(a_m)\, da_m = \frac{\phi(0)\chi(\varpi) \cdot q^{-ms}}{1 - \chi(\varpi) \cdot q^{-ms}} \int_{\mathcal{O}^*} \chi(a_m)\, da_m.$$

Regarding $i = 0$: the function $F_\phi(\varepsilon a_m k') \chi(a_m)$ of the variable a_m is constant on cosets of $1 + \varpi\mathcal{O}$, and therefore

$$\int_{\mathcal{O}^*} F_\phi(\varepsilon a_m k') \chi(a_m)\, da_m = \int_{(1+\varpi\mathcal{O})\backslash^{\mathcal{O}^*}} F_\phi(\varepsilon a k') \chi(a)\, da$$

Since ${}_{(1+\varpi\mathcal{O})}\backslash^{\mathcal{O}^*} \cong \mathbb{F}_q^*$ by ν we get

$$\int_{\mathcal{O}^*} F_\phi(\varepsilon a_m k') \chi(a_m)\, da_m = \frac{1}{\left|\mathbb{F}_q^*\right|} \sum_{a \in \mathbb{F}_q^*} \phi(\varepsilon a \cdot \nu(k')) \omega_{\pi_0}(a).$$

Therefore, we are left with the integral

$$J_{\pi,\psi}(s, W_v, F_\phi) = \int_K dk' \int_{\mathcal{N}^-(\mathcal{O})} dX \left(W_v \left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} k' & \\ & k' \end{pmatrix} \right) \right) \cdot$$
$$\left(\frac{1}{\left|\mathbb{F}_q^*\right|} \sum_{a \in \mathbb{F}_q^*} \phi(\varepsilon a \cdot \nu(k')) \omega_{\pi_0}(a) + \frac{\phi(0)\chi(\varpi) \cdot q^{-ms}}{1 - \chi(\varpi) \cdot q^{-ms}} \int_{\mathcal{O}^*} \chi(a_m)\, da_m \right).$$

Since $W_v \restriction_{\mathrm{GL}_{2m}(\mathcal{O})} = W_v^0 \circ \nu$, the integrand is constant in the variable k' on cosets of ${}_{(I_m + \varpi M_m(\mathcal{O}))}\backslash^{\mathrm{GL}_m(\mathcal{O})}$, and is constant in the variable X on cosets of ${}_{\varpi\mathcal{N}^-(\mathcal{O})}\backslash^{\mathcal{N}^-(\mathcal{O})}$, and therefore

$$J_{\pi,\psi}(s, W_v, F_\phi) = \int_{(I_m + \varpi M_m(\mathcal{O}))\backslash^{\mathrm{GL}_m(\mathcal{O})}} dk' \int_{\varpi\mathcal{N}^-(\mathcal{O})\backslash^{\mathcal{N}^-(\mathcal{O})}} dX \left(W_v \left(w_{m,m} \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \begin{pmatrix} k' & \\ & k' \end{pmatrix} \right) \right) \cdot$$
$$\left(\frac{1}{\left|\mathbb{F}_q^*\right|} \sum_{a \in \mathbb{F}_q^*} \phi(\varepsilon a \cdot \nu(k')) \omega_{\pi_0}(a) + \frac{\phi(0)\chi(\varpi) \cdot q^{-ms}}{1 - \chi(\varpi) \cdot q^{-ms}} \int_{\mathcal{O}^*} \chi(a_m)\, da_m \right).$$

Since we have the following isomorphisms (by the map ν):

$$ {}_{(I_m+\varpi M_m(\mathcal{O}))}\backslash^{\mathrm{GL}_m(\mathcal{O})} \cong \mathrm{GL}_m\left(\mathbb{F}_q\right), $$
$$ {}_{\varpi\mathcal{N}^-(\mathcal{O})}\backslash^{\mathcal{N}^-(\mathcal{O})} \cong \mathcal{N}^-\left(\mathbb{F}_q\right), $$

we get

$$ J_{\pi,\psi}\left(s, W_v, F_\phi\right) = \frac{1}{\left|\mathrm{GL}_m\left(\mathbb{F}_q\right)\right|}\frac{1}{\left|\mathcal{N}^-\left(\mathbb{F}_q\right)\right|}\sum_{k'\in\mathrm{GL}_m(\mathbb{F}_q)}\sum_{X\in\mathcal{N}^-(\mathbb{F}_q)}\left(W_v^0\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m\end{pmatrix}\begin{pmatrix} k' & \\ & k'\end{pmatrix}\right)\right)\cdot $$
$$ \left(\frac{1}{\left|\mathbb{F}_q^*\right|}\sum_{a\in\mathbb{F}_q^*}\phi\left(\varepsilon a\cdot k'\right)\omega_{\pi_0}\left(a\right) + \frac{\phi\left(0\right)\chi\left(\varpi\right)\cdot q^{-ms}}{1-\chi\left(\varpi\right)\cdot q^{-ms}}\int_{\mathcal{O}^*}\chi\left(a_m\right)da_m\right). $$

Note that for a fixed $X\in\mathcal{N}^-\left(\mathbb{F}_q\right)$, replacing ak' with k', and using the fact that ω_{π_0} is the central character of π_0 yields

$$ \frac{1}{\left|\mathbb{F}_q^*\right|}\sum_{a\in\mathbb{F}_q^*}\sum_{k'\in\mathrm{GL}_m(\mathbb{F}_q)}\left(W_v^0\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m\end{pmatrix}\begin{pmatrix} k' & \\ & k'\end{pmatrix}\right)\right)\phi\left(\varepsilon a\cdot k'\right)\omega_{\pi_0}\left(a\right) = $$
$$ \sum_{k'\in\mathrm{GL}_m(\mathbb{F}_q)}\left(W_v^0\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m\end{pmatrix}\begin{pmatrix} k' & \\ & k'\end{pmatrix}\right)\right)\phi\left(\varepsilon k'\right). $$

For $X\in\mathcal{N}^-\left(\mathbb{F}_q\right)$, we have $\mathrm{tr}X=0$ and therefore

$$ J_{\pi,\psi}\left(s, W_v, F_\phi\right) = \frac{1}{\left|\mathrm{GL}_m\left(\mathbb{F}_q\right)\right|}\frac{1}{\left|\mathcal{N}^-\left(\mathbb{F}_q\right)\right|}\sum_{k'\in\mathrm{GL}_m(\mathbb{F}_q)}\sum_{X\in\mathcal{N}^-(\mathbb{F}_q)}\left(W_v^0\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m\end{pmatrix}\begin{pmatrix} k' & \\ & k'\end{pmatrix}\right)\right)\cdot $$
$$ \cdot\,\psi_0\left(-\mathrm{tr}X\right)\cdot\left(\phi\left(\varepsilon k'\right) + \frac{\phi\left(0\right)\chi\left(\varpi\right)\cdot q^{-ms}}{1-\chi\left(\varpi\right)\cdot q^{-ms}}\int_{\mathcal{O}^*}\chi\left(a_m\right)da_m\right), $$

We have shown that this summand is constant in the variable k' on cosets of ${}_{N_m(\mathbb{F}_q)}\backslash^{\mathrm{GL}_m(\mathbb{F}_q)}$ and constant in the variable X on cosets of ${}_{\mathcal{B}(\mathbb{F}_q)}\backslash^{M_m(\mathbb{F}_q)} \cong \mathcal{N}^-\left(\mathbb{F}_q\right)$ (Proposition 1.8). Using these observations, we get

$$ J_{\pi,\psi}\left(s, W_v, F_\phi\right) = J_{\pi_0,\psi_0}\left(W_v^0,\phi\right) + J_{\pi_0,\psi_0}\left(W_v^0,1\right)\cdot\frac{\phi\left(0\right)\chi\left(\varpi\right)\cdot q^{-ms}}{1-\chi\left(\varpi\right)\cdot q^{-ms}}\int_{\mathcal{O}^*}\chi\left(a_m\right)da_m. $$

Finally, notice that if $J_{\pi_0,\psi_0}\left(W_v^0,1\right)\neq 0$, for some $v\in V_0$, then W_v^0 defines a Shalika vector (See also Proposition 2.13), and therefore $\omega_{\pi_0}\equiv 1$ and $\int_{\mathcal{O}^*}\chi\left(a_m\right)da_m = 1$. Otherwise, $J_{\pi_0,\psi_0}\left(W_v^0,1\right)=0$, for every $v\in V_0$. In both cases we get that

$$ J_{\pi,\psi}\left(s, W_v, F_\phi\right) = J_{\pi_0,\psi_0}\left(W_v^0,\phi\right) + J_{\pi_0,\psi_0}\left(W_v^0,1\right)\cdot\frac{\phi\left(0\right)\chi\left(\varpi\right)\cdot q^{-ms}}{1-\chi\left(\varpi\right)\cdot q^{-ms}}, $$

as required. □

Repeating the same steps for the expression

$$ \tilde{J}_{\pi,\psi}\left(s, W_v, \phi\right) = \int_{N\backslash G}\int_{\mathcal{B}\backslash M} W\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m\end{pmatrix}\begin{pmatrix} g & \\ & g\end{pmatrix}\right)\psi\left(-\mathrm{tr}X\right)dX\cdot\hat{\phi}\left(\varepsilon_1 g^l\right)\left|\det g\right|^{s-1}dg, $$

with the same Haar measures, and using the fact that $\widehat{F_\phi} = F_{\hat{\phi}}$ yields

Proposition 4.8. *For any $\phi : \mathbb{F}_q^m \to \mathbb{C}$ and $v \in V_0$ one has*

$$\tilde{J}_{\pi,\psi}\left(s, W_v, F_\phi\right) = \tilde{J}_{\pi_0,\psi_0}\left(W_v^0, \phi\right) + J_{\pi_0,\psi_0}\left(W_v^0, 1\right) \cdot \frac{\hat{\phi}\left(0\right) \chi^{-1}\left(\varpi\right) \cdot q^{-m(1-s)}}{1 - \chi^{-1}\left(\varpi\right) \cdot q^{-m(1-s)}}.$$

Proof. We specify only the modifications that need to be done for the dual Jacquet-Shalika integral. One begins with

$$\tilde{J}_{\pi,\psi}\left(s, W_v, F_\phi\right) = \int_{A_{m-1}} da' \int_K dk \int_{\mathcal{N}^-} dX \left(\delta_B^{-1}\left(a'\right) W_v\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m \end{pmatrix}\begin{pmatrix} a'k & \\ & a'k \end{pmatrix}\right)\right) \left|\det\left(a'\right)\right|^{s-1} \cdot$$
$$\cdot \int_{F^*} F_{\hat{\phi}}\left(\varepsilon_1 a_1^{-1} k^l\right) \left|a_1\right|^{m(1-s)} \omega_\pi^{-1}\left(a_1\right) da_1.$$

This expression is obtained by beginning with the Iwasawa decomposition and substituting $a = a_1^{-1} \cdot a'$, where this time we think of $A_{m-1} \subseteq A_m$ by the embedding $\operatorname{diag}\left(a_2', \dots, a_m'\right) \mapsto \operatorname{diag}\left(1, a_2', , \dots, a_m'\right)$. Proceeding using the same steps as in the proof of Theorem 3.23, we arrive to the expression

$$\tilde{J}_{\pi,\psi}\left(s, W_v, F_\phi\right) = \int_{A_{m-1}} da' \int_K dk \int_{\mathcal{N}^-} dZ \left(\delta_B^{-2}\left(a'\right) \psi\left(b n_Z b^{-1}\right) W_v\left(b t_Z k_Z w_{m,m}\begin{pmatrix} k & \\ & k \end{pmatrix}\right)\right) \left|\det\left(a'\right)\right|^{s-1} \cdot$$
$$\cdot \int_{F^*} F_{\hat{\phi}}\left(\varepsilon_1 a_1^{-1} k^l\right) \left|a_1\right|^{m(1-s)} \omega_\pi^{-1}\left(a_1\right) da_1,$$

where $Z = a'^{-1} X a'$, $b = \operatorname{diag}\left(1, 1, a_2', a_2', \dots, a_{m-1}', a_{m-1}', a_m', a_m'\right)$, $u_Z = \left(\begin{smallmatrix} I_m & Z \\ & I_m \end{smallmatrix}\right)$ and $u_Z = n_Z t_Z k_Z$ is an Iwasawa decomposition of u_Z as in Proposition 3.22.

One proceeds as in the previous proof, but this time uses the fact that if $t_Z = \operatorname{diag}\left(t_1, t_2, \dots, t_{2m-1}, t_{2m}\right)$, then $|t_1| = 1$ (Theorem 3.15).

After showing that the integral is integrated on $a' \in \left(\mathcal{O}^*\right)^{m-1}$, $Z \in \mathcal{N}^-\left(\mathcal{O}\right)$, one notices that $\varepsilon_1 a_1^{-1}\left(a'^{-1}\right)^l \left(k'\right)^l = \varepsilon_1 a_1^{-1}\left(k'\right)^l$, as the first row of a' is ε_1.

The rest of the proof is similar to the previous proof. ☐

Corollary 4.9. *Suppose that π_0 does not admit a Shalika vector. Then $\gamma_{\pi,\psi}\left(s\right) = \gamma_{\pi_0,\psi_0}$.*

Proof. By Proposition 2.13, π_0 admits a Shalika vector if and only if $J_{\pi_0,\psi_0}\left(W_v^0, 1\right) \neq 0$, for some $v \in V_0$. Therefore if π_0 does not admit a Shalika vector, then $J_{\pi,\psi}\left(s, W_v, F_\phi\right) = J_{\pi_0,\psi_0}\left(W_v^0, \phi\right)$ and $\tilde{J}_{\pi,\psi}\left(s, W_v, F_\phi\right) = \tilde{J}_{\pi_0,\psi_0}\left(W_v^0, \phi\right)$, and therefore $\gamma_{\pi,\psi}\left(s\right) = \gamma_{\pi_0,\psi_0}$. ☐

4.3. **The γ-factor of a level zero supercuspidal representation admitting a Shalika vector.** As in the previous subsection, let π_0 be an irreducible cuspidal representation of $\mathrm{GL}_{2m}\left(\mathbb{F}_q\right)$ and let π be a level zero supercuspidal representation, constructed through π_0, with respect to the central character $\chi : F^* \to \mathbb{C}$. In this subsection, we assume that π_0 admits a Shalika vector, and compute the γ-factor of π.

Suppose that $v \in V_0$, such that $J_{\pi_0,\psi_0}\left(W_v^0, 1\right) = 1$. We choose $\phi\left(x\right) = \delta_0\left(x\right) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$.

Then $\hat{\phi}\left(x\right) = \frac{1}{q^m}$, and we have

$$J_{\pi,\psi}\left(s, W_v, F_\phi\right) = \frac{\chi\left(\varpi\right) \cdot q^{-ms}}{1 - \chi\left(\varpi\right) \cdot q^{-ms}} = \chi\left(\varpi\right) \cdot q^{-ms} L\left(ms, \chi\right).$$

Since $\tilde{J}_{\pi_0,\psi_0}(W_v^0,\phi) = \frac{1}{q^m} J_{\pi_0,\psi_0}(W_v^0,1) = \frac{1}{q^m}$, we have

$$\tilde{J}_{\pi,\psi}(s,W_v,F_\phi) = \frac{1}{q^m}\frac{1}{1-\chi^{-1}(\varpi)\cdot q^{-m(1-s)}} = q^{-m} L\left(m(1-s),\chi^{-1}\right).$$

It follows that

$$\gamma_{\pi,\psi}(s) = \frac{q^{ms}}{q^m \chi(\varpi)} \cdot \frac{L(m(1-s),\chi^{-1})}{L(ms,\chi)}.$$

By choosing $\phi = 1$, it is clear that $L(s,\pi,\wedge^2) = L(ms,\chi)$, and that $L(s,\tilde{\pi},\wedge^2) = L(ms,\chi^{-1})$. Therefore $\varepsilon_{\pi,\psi}(s) = \frac{q^{ms}}{q^m\chi(\varpi)}$.

4.4. **The modified functional equation.** Using the results of the previous subsections, we obtain a modified functional equation for the Jacquet-Shalika integral over a finite field.

Unlike the functional equation presented in Subsection 2.3 (Theorem 2.6), the modified equation is valid for all irreducible cuspidal representations of $\mathrm{GL}_{2m}(\mathbb{F}_q)$, regardless whether they admit a Shalika vector or not.

Let $\psi : \mathbb{F}_q \to \mathbb{C}^*$ be a non-trivial character of $\mathbb{F}_q$.

Theorem 4.10. *Let π be an irreducible cuspidal representation of $\mathrm{GL}_{2m}(\mathbb{F}_q)$. Then there exists a rational function $\gamma_{\pi,\psi}(s) \in \mathbb{C}(q^{-s})$, such that for every $s \in \mathbb{C}$, $W \in \mathcal{W}(\pi,\psi)$, $\phi \in \mathcal{S}\left(\mathbb{F}_q^m\right)$, one has*

$$\gamma_{\pi,\psi}(s)\left(J_{\pi,\psi}(W,\phi) + J_{\pi,\psi}(W,1)\cdot\phi(0)\,q^{-ms}L(ms,1)\right) =$$
$$\tilde{J}_{\pi,\psi}(W,\phi) + J_{\pi,\psi}(W,1)\cdot\hat{\phi}(0)\,q^{-m(1-s)}L(m(1-s),1).$$

From this equation alone, one can easily see that if π does not admit a Shalika vector, then $\gamma_{\pi,\psi}(s) \in \mathbb{C}^*$, and otherwise

$$\gamma_{\pi,\psi}(s) = \frac{q^{ms}}{q^m}\cdot\frac{L(m(1-s),1)}{L(ms,1)}.$$

To show this, one uses Proposition 2.13. If π doesn't admit a Shalika vector, then $J_{\pi,\psi}(W,1) = 0$, for every $W \in \mathcal{W}(\pi,\psi)$, and we get the same functional equation as in Theorem 2.6. If π admits a Shalika vector, then there exists $W_0 \in \mathcal{W}(\pi,\psi)$ such that $J_{\pi,\psi}(W_0,1) = 1$. One substitutes $W = W_0$, $\phi = \delta_0$, as in the previous subsection, to get the above form of $\gamma_{\pi,\psi}$.

Thus the modified functional equation relates between a pole of $\gamma_{\pi,\psi}(s)$ and the existence of a Shalika vector of π in a simple matter.

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קישור בין התורות אנו מסיימים את העבודה ע"י קישור בין התורה של האינטגרל של Jacquet וShalika מעל שדה סופי ומעל שדה p־אדי, באמצעות הצגות חוד מסוג level zero (depth zero). התוצאות העיקריות שלנו הן המשפטים הבאים:

משפט (H). תהי (π_0, V_{π_0}) הצגת חוד אי־פריקה של $\mathrm{GL}_{2m}(\mathbb{F}_q)$, ותהי π הצגת חוד מסוג level zero, שנבנתה דרך π_0. אז לכל $v \in V_{\pi_0}$, $\phi \in \mathcal{S}(\mathbb{F}_q^m)$ ו$s \in \mathbb{C}$, מתקיים

$$J_{\pi,\psi}(s, W_v, F_\phi) = J_{\pi_0,\psi_0}\left(W_v^0, \phi\right) + J_{\pi_0,\psi_0}\left(W_v^0, 1\right) \cdot \phi(0)\, \omega_\pi(\varpi) \cdot q^{-ms} L(ms, \omega_\pi).$$

כמסקנה, אנו מקבלים את הגרסה הבאה של המשוואה הפונקציונלית, שכעת נכונה לכל הצגת חוד אי־פריקה π של $\mathrm{GL}_{2m}(\mathbb{F}_q)$, ללא תלות בתנאי האם יש לπ וקטור Shalika.

משפט (D"). קיים איבר $\gamma_{\pi,\psi}(s) \in \mathbb{C}(q^{-s})$, כך שלכל $W \in \mathcal{W}(\pi, \psi)$, $\phi \in \mathcal{S}(\mathbb{F}_q^m)$ ו$s \in \mathbb{C}$, מתקיים

$$J_{\tilde{\pi},\psi^{-1}}\left(\tilde{\pi}\left(\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}\right)\tilde{W}, \hat{\phi}\right) + J_{\pi,\psi}(W, 1) \cdot \hat{\phi}(0) \cdot q^{-m(1-s)} L(m(1-s), 1) = \gamma_{\pi,\psi}(s) \cdot \left(J_{\pi,\psi}(W, \phi) + J_{\pi,\psi}(W, 1) \cdot \phi(0) \cdot q^{-ms} L(ms, 1)\right).$$

יתר על כן, אם לπ יש וקטור Shalika, אז

$$\gamma_{\pi,\psi}(s) = \frac{q^{ms}}{q^m} \frac{L(m(1-s), 1)}{L(ms, 1)}.$$

אחרת, $\gamma_{\pi,\psi} \in \mathbb{C}^*$.

התורה מעל שדה סופי. בנוסף, אנחנו מפתחים תורה אנלוגית לאינטגרל של Jacquet וShalika מעל שדה סופי $\mathbb{F}_q$. כעת נפרט את התוצאות העיקריות שלנו.

תהי π הצגה אי־פריקה גנרית של $\mathrm{GL}_{2m}\left(\mathbb{F}_q\right)$.

משפט ('B). קיימים $W \in \mathcal{W}\left(\pi,\psi\right)$, $\phi \in \mathcal{S}\left(\mathbb{F}_q^m\right)$, כך ש

$$1 = J_{\pi,\psi}\left(W,\phi\right) = \frac{1}{\left[\mathrm{GL}_m\left(\mathbb{F}_q\right):N\right]\left[M_m\left(\mathbb{F}_q\right):\mathcal{B}\right]} \sum_{g\in N\backslash^{\mathrm{GL}_m\left(\mathbb{F}_q\right)}} \sum_{X\in\mathcal{B}\backslash^{M_m\left(\mathbb{F}_q\right)}} W\left(w_{m,m}\begin{pmatrix} I_m & X \\ & I_m\end{pmatrix}\begin{pmatrix} g & \\ & g\end{pmatrix}\right) \cdot \psi\left(-\mathrm{tr}X\right)\cdot\phi\left(\varepsilon g\right).$$

נניח מעתה כי π הצגת חוד.

משפט ('D). נניח כי לπ אין וקטור Shalika. אז קיים קבוע $\gamma_{\pi,\psi} \in \mathbb{C}^*$, כך שלכל $W \in \mathcal{W}\left(\pi,\psi\right)$ ו$\phi \in \mathcal{S}\left(\mathbb{F}_q^m\right)$, מתקיים

$$\gamma_{\pi,\psi}\cdot J_{\pi,\psi}\left(W,\phi\right) = J_{\tilde{\pi},\psi^{-1}}\left(\tilde{\pi}\left(\begin{pmatrix} & I_m \\ I_m & \end{pmatrix}\right)\tilde{W},\hat{\phi}\right).$$

יהי $\theta:\mathbb{F}_{q^{2m}}^*\to\mathbb{C}^*$ כרקטר רגולרי המתאים לπ.

משפט ('E). התנאים הבאים שקולים:

1. קיים $0\neq W\in\mathcal{W}\left(\pi,\psi\right)$, כך ש $J_{\pi,\psi}\left(W,\phi\right)\neq 0$.
2. לπ יש וקטור Shalika.
3. $\theta\restriction_{\mathbb{F}_{q^m}^*}\equiv 1$.

אנחנו מבטאים את $\gamma_{\pi,\psi}$ באמצעות הכרקטר θ, עבור המקרים $m=1,2$.

משפט (G). נניח כי $\theta\restriction_{\mathbb{F}_{q^m}^*}\not\equiv 1$ (כלומר לπ אין וקטור Shalika). אז

1. ל$m=1$,

$$\gamma_{\pi,\psi}^{-1} = \sum_{a\in\mathbb{F}_q^*}\omega_\pi\left(a\right).\psi^{\mathcal{F}}\left(-a\right).$$

2. ל$m=2$,

$$\gamma_{\pi,\psi}^{-1} = T_0 - \frac{1}{q^2}\left(\sum_{a\in\mathbb{F}_q^*}\omega_\pi\left(a\right)\psi^{\mathcal{F}}\left(-a\right)\right)\left(\sum_{b\in\mathbb{F}_q^*}\left(\sum_{\substack{\xi\in\mathbb{F}_{q^4}^*\\ \mathrm{N}_{\mathbb{F}_{q^4}/\mathbb{F}_q}(\xi)=b^2}}\sum_{\beta\in\mathbb{F}_q^*}\psi^{-1}\left(\beta+\frac{1}{\beta}\mathrm{Tr}_{\mathbb{F}_{q^4}/\mathbb{F}_q}\left(\xi+\frac{b}{\xi}\right)\right)\theta\left(\xi\right)\right)\right),$$

כאשר $T_0 = \begin{cases} q-\frac{1}{q} & \omega_\pi\equiv 1 \\ 0 & \omega_\pi\not\equiv 1\end{cases}$.

במאמרם [KR12], Kewat וRaghunathan מסמנים $L_{JS}\left(s,\pi,\wedge^2\right)=\frac{1}{p(q^{-s})}$, ומראים שלכל הצגה אי־פריקה גנרית חלקה של $\mathrm{GL}_{2m}\left(F\right)$, $L_{JS}\left(s,\pi,\wedge^2\right)$ היא אותה הפונקציה כמו זו שנבנתה מעלה ע"י התאמת Langlands (Jacquet וShalika מראים זאת רק להצגות לא־מסועפות).

כתוצאה ממשפט C, ל$J_{\pi,\psi}\left(s,W,\phi\right)$ יש המשכה מרומורפית לכל המישור, אותה אנו ממשיכים לסמן ב$J_{\pi,\psi}\left(s,W,\phi\right)$.

נניח מעתה כי π היא הצגת חוד. אנו מוכיחים את המשפטים הבאים.

משפט (D). קיים איבר $\gamma_{\pi,\psi}\left(s\right)\in\mathbb{C}\left(q^{-s}\right)$, כך שלכל $\phi\in\mathcal{S}\left(F^m\right)$, $W\in\mathcal{W}\left(\pi,\psi\right)$, מתקיים

$$J_{\tilde{\pi},\psi^{-1}}\left(1-s,\tilde{\pi}\left(\begin{pmatrix} & I_m\\ I_m & \end{pmatrix}\right)\tilde{W},\hat{\phi}\right)=\gamma_{\pi,\psi}\left(s\right)\cdot J_{\pi,\psi}\left(s,W,\phi\right).$$

יתר על כן,

$$\gamma_{\pi,\psi}\left(s\right)=\varepsilon_{\pi,\psi}\left(s\right)\cdot\frac{L\left(1-s,\tilde{\pi},\wedge^2\right)}{L\left(s,\pi,\wedge^2\right)},$$

כאשר $\varepsilon_{\pi,\psi}\left(s\right)$ איבר הפיך של $\mathbb{C}\left[q^{-s},q^{s}\right]$.

אנו עוקבים אחר ההוכחה של Matringe [Mat12, Mat14] כדי להוכיח את משפט D.

משפט (E). התנאים הבאים שקולים:

1. $\omega_\pi\equiv 1$ וקיים $W\in\mathcal{W}\left(\pi,\psi\right)$, כך ש

$$l_{\pi,\psi}\left(W\right)=\int_{ZN\backslash\mathrm{GL}_m(F)}\int_{\mathcal{B}\backslash M_m(F)}W\left(w_{m,m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}g & \\ & g\end{pmatrix}\right)\psi\left(-\mathrm{tr}X\right)dXdg\neq 0.$$

2. ל$\gamma_{\pi,\psi}\left(s\right)$ יש קוטב ב$s=1$.
3. ל$L\left(s,\pi,\wedge^2\right)$ יש קוטב ב$s=0$.

אנחנו מוכיחים את משפט E דרך המשוואה הפונקציונלית שנידונה במשפט D. משפט דומה כבר ידוע לפונקציית L של $\wedge^2$ המגיעה מהבניה של Shahidi (ראו גם את ההקדמה של [JNQ08] ואת משפט 5.5 של המאמר הנ"ל).

משפט (F). אם ω_π הוא מסועף, אז $L\left(s,\pi,\wedge^2\right)=L\left(ms,\omega_\pi\right)=1$. אם ω_π הוא לא־מסועף, אז

$$L\left(s,\pi,\wedge^2\right)=\prod_{k\in S_{\pi,\psi}}\frac{1}{1-\omega_\pi\left(\varpi\right)^{\frac{1}{m}}\zeta^k q^{-s}},$$

כאשר $\zeta=e^{\frac{2\pi i}{m}}$ ו

$$\begin{aligned}S_{\pi,\psi}=&\Big\{0\leq k\leq m-1\mid\exists W\in\mathcal{W}\left(\pi,\psi\right),\\ &\int_{ZN\backslash G}\left(\int_{\mathcal{B}\backslash M}W\left(w_{m,m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}g & \\ & g\end{pmatrix}\right)\psi\left(-\mathrm{tr}\left(X\right)\right)dX\right)\left|\det g\right|^{\frac{2\pi ik-\log\omega_\pi(\varpi)}{m\log q}}dg\neq 0\Big\}.\end{aligned}$$

הקדמה

יהי F שדה מקומי לא־ארכימדי. תהי π הצגה חלקה אי־פריקה של $\mathrm{GL}_n(F)$. מהתאמת Langlands המקומית, קיימת הצגה $\rho(\pi)$ המתאימה ל־π, של חבורת Weil-Delinge W'_F. הצגה זו היא ממימד n. פונקציית L המקומית של $\wedge^2$ המתאימה ל־π, מוגדרת דרך התאמה זו ע"י $L(s,\pi,\wedge^2)=L(s,\wedge^2(\rho(\pi)))$. אנו נתעניין רק במקרה בו n זוגי.

במאמרם [JS90], Jacquet וShalika חוקרים את פונקציית L הגלובלית של $\wedge^2$ המתאימה להצגות חוד אוטומורפיות אי־פריקות של GL_n, בעיקר עבור המקרה בו n זוגי. בחלק 7 של [JS90], Jacquet וShalika נותנים הצגה אינטגרלית ל־$L(s,\pi,\wedge^2)$, להצגות אי־פריקות לא מסועפות של $\mathrm{GL}_{2m}(F)$. מצד שני, במאמרו [Sha90] בחלק 7, Shahidi מציע בניה פוטנציאלית נוספת לפונקציית L המקומית של $\wedge^2$, דרך הLanglands-Shahidi method. במאמרם [KR12], Kewat וRaghunathan מראים ששלוש בניות אלה לפונקציית L המקומית של $\wedge^2$ מסכימות, לכל ההצגות החלקות האי־פריקות הגנריות של $\mathrm{GL}_{2m}(F)$ [KR12, Theorem 1.4].

במאמרו [Mat14], Matringe מוכיח את המשוואה הפונקציונלית המקומית המתאימה. משוואה זו מוכחת כבר ע"י Kewat וRaghunathan במאמר [KR12], ע"י שימוש בארגומנטים גלובליים. ההוכחה של Matringe עושה שימוש רק בארגומנטים מקומיים.

עבודה זו עוסקת בתורה הלא־ארכימדית של האינטגרל של Jacquet וShalika שהוזכר לעיל. במשפטים A-D להלן, אנו נותנים סקירה אודות התוצאות הידועות של תורה זו. אנו עוקבים אחר ההוכחות של Jacquet וShalika ושל Matringe, ומוסיפים פרטים להוכחות המקוריות. התרומה שלנו היא התורות והמשפטים המופיעים לאחר משפט D, למרות שייתכן שאלה ידועים למומחים בתחום.

נציג כעת את המשפטים העיקריים שנוכיח.

התורה מעל שדה p־אדי. יהי F שדה p־אדי. תהי π הצגה חלקה אי־פריקה גנרית של $\mathrm{GL}_{2m}(F)$.

משפט (A). קיים $r_{\pi,\wedge^2}\in\mathbb{R}$, כך שלכל $s\in\mathbb{C}$ עם $\mathrm{Re}(s)>r_{\pi,\wedge^2}$, $W\in\mathcal{W}(\pi,\psi)$, $\phi\in\mathcal{S}(F^m)$, האינטגרל הבא מתכנס בהחלט

$$J_{\pi,\psi}(s,W,\phi)=\int_{N\backslash\mathrm{GL}_m(F)}\int_{\mathcal{B}\backslash M_m(F)}W\left(w_{m,m}\begin{pmatrix}I_m & X\\ & I_m\end{pmatrix}\begin{pmatrix}g & \\ & g\end{pmatrix}\right)\psi(-\mathrm{tr}X)\,dX\cdot\phi(\varepsilon g)\left|\det g\right|^s dg.$$

משפט (B). קיימים $W\in\mathcal{W}(\pi,\psi)$, $\phi\in\mathcal{S}(F^m)$, כך שלכל $s\in\mathbb{C}$ עם $\mathrm{Re}(s)>r_{\pi,\wedge^2}$,

$$J_{\pi,\psi}(s,W,\phi)=1.$$

אנו עוקבים אחר ההוכחות של Jacquet וShalika [JS90, Sections 7.1, 7.3] כדי להוכיח את משפטים A וB.

משפט (C). עבור $W\in\mathcal{W}(\pi,\psi)$, $\phi\in\mathcal{S}(F^m)$ קבועים, הפונקציה $J_{\pi,\psi}(s,W,\phi)$ היא איבר של $\mathbb{C}(q^{-s})$, ל־s בתחום ההתכנסות, ולכן יש לה המשכה מרומורפית לכל המישור. יותר על כן, נסמן

$$I_{\pi,\psi}=\mathrm{span}_{\mathbb{C}}\left\{J_{\pi,\psi}(s,W,\phi)\mid W\in\mathcal{W}(\pi,\psi),\,\phi\in\mathcal{S}(F^m)\right\},$$

אז קיים איבר יחיד $p(z)\in\mathbb{C}[z]$, כך ש־$p(0)=1$ ו־$I_{\pi,\psi}=\frac{1}{p(q^{-s})}\mathbb{C}[q^{-s},q^s]$. מסמנים $L(s,\pi,\wedge^2)=\frac{1}{p(q^{-s})}$.

הפקלוטה למדעים מדויקים
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על פונקציות גמא $\gamma_{\pi,\psi,\wedge^2}(s)$ של הצגות של GL_{2m}

חיבור זה מוגש כחלק מהדרישות לקבלת תואר "מוסמך אוניברסיטה" בבית הספר למדעי המתמטיקה, אוניברסיטת תל אביב

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